The fortress and prison yard problems in arbitrary 2-manifolds

Lawrence Erickson

November 3, 2009

A closed polygonal curve in the plane divides the plane into two regions. Call the open region inside the curve E, the open region outside the curve E', and the curve itself ∂E. Point a is *visible* from point b if the line segment $\overline{ab}$ does not intersect both E and E'. The art gallery problem is a famous problem that asks, given a polygonal region E, what is the minimum size of a set of "guard points" G such that every $e \in E$ is visible from some $g \in G$. It was shown in [2] that $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary for an n vertex polygon, and [1] describes an algorithm that determines the proper locations of the guards.

A related question is the "fortress problem". In the fortress problem, the goal is instead to minimize a set of guard points $G \subseteq \partial E$ such that for every $e \in E'$ (the exterior), e is visible from some $g \in G$. In [4], it was shown that the fortress problem takes at most $\lceil n/2 \rceil$ guards to cover the exterior.

Suppose that, instead of being embedded in a plane, the polygon were instead embedded in a fundamental polygon of some two-manifold that had the plane as its covering space, like a torus (see Figure 1). Clearly, since one guard can "see" over the line where the face connects to itself, fewer guards are needed, but it is not clear what the exact number would be. It seems as though a bound of $\lfloor n/3 \rfloor$ should suffice, as the "exterior" is just the interior of another polygonally bounded region. Also, there may be more than one straight line from one point to another.

Finally, there is the prison yard problem, which is a sort of combination of the previous two problems. The prison yard problem once again is about a polygon embedded in the plane. However, this polygon can be multiply connected, splitting the plane into sets $E_1, E_2, \ldots, E_k$. A point a is visible from a point b if the line segment $\overline{ab}$ does not intersect more than one region E_i. In this problem, the goal is to find a set of guards $G \subseteq \partial E$ such that for each $e \in \bigcup_{i=1}^{k} E_i$, there exists a $g \in G$ such that e is visible from g. For a single simple polygon, the bound was proven in [3] to be $\lceil n/2 \rceil$ for convex polygons, and $\lfloor n/2 \rfloor$ for non-convex polygons.

Again, it is obvious that embedding this in a fundamental polygon of some two-manifold changes the required number of guards somewhat. Due to the identification of some sides of the polygon together, certain visibility rays may

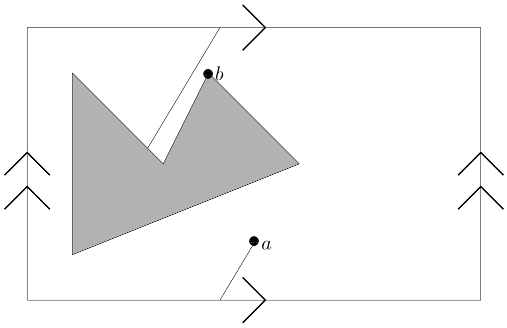

Figure 1: Because the sides are identified, the point a can see some portions of the polygon's exterior that would be hidden if the polygon were embedded in the plane. A guard at point b can see the entire exterior.

appear to be space-filling curves. Therefore, it is possible that solutions to [5], which describes algorithms for finding billiards-like space-filling curves on a number of different surfaces.

So, what are the bounds required for polygons embedded on fundamental polygons of various two-manifolds on both the fortress and prison yard problems, and how does one compute the optimal guard placements?

References

[1] D. Avis and G. T. Toussaint, "An optimal algorithm for determining the visibility of a polygon from an edge," *IEEE Transactions on Computation*, vol. C-30, pp. 910–914, 1981.

[2] V. Chvatal, "A combinatorial theorem in plane geometry," *Journal of Combinatorial Theory Ser. B*, vol. 18, pp. 39–41, 1975.

[3] Z. Furedi and D. J. Kleitman, "The prison yard problem," *Combinatorica*, vol. 14, pp. 287–300, 1994.

[4] J. O'Rourke, *Art Gallery Theorems and Algorithms.* Cambridge, UK: Oxford University Press, 1987.

[5] S. Ramamoorthy, R. Rajagopal, Q. Ruan, and L. Wenzel, "Low-discrepancy curves and efficient coverage of space," in *Proc. Workshop on the Algorithmic Foundations of Robotics*, New York, July 2006.