# Crossing Numbers

Matthew Yancey
yancey1@illinois.edu

October 11, 2009

A simple graph is a set of vertices $V$ and edges $E \subset \binom{V}{2}$. Many problems revolve around the idea of embedding the graph as the subspace of an ambient space, usually $\mathbb{R}^2$. Considering ambient spaces with larger fundamental groups is an important area of combinatorial topology. Let us only consider simple, finite graphs.

## 1 General Topological Concepts

An embedding is considered proper if an edge only vertices it touches are the endpoints. If no edges are incident to other edges, then the embedding is called planar. If it is possible for a graph $G$ to have a planar embedding in $\mathbb{R}^2$, then the graph is called planar.

One generalization of this is to consider the crossing number, $cr(G)$, which equals the minimum number of times the edges of $G$ are incident to each other, across all embeddings of $G$. For all planar graphs, $cr(G) = 0$. Also, $cr(K_5) = cr(K_{3,3}) = 1$.

The crossing number is a parameter of a graph that has been studied for decades. Theorems exist placing boundaries on it using other graph parameters: $|V|$, $|E|$, etc [1]. They also find exact values or more precise bounds for specific families of graphs such as $K_n$, $K_{m,n}$, and $C_m \otimes C_n$.

## 2 General Algorithmic Results

Finding algorithms to determine the crossing number of a graph has been a problem of interest lately. It has frequently been considered as a generalization to the problem of determining whether or not a graph is planar. A different extension to the same problem is finding the minimum genus of a surface on which the graph can be embedded planarly. Many results in the crossing number problem use ideas from the planar and genus problem.

The genus problem was found to be NP-hard, but linear-time algorithms exist to determine if a graph will embed in an arbitrary surface if the surface is fixed [4]. The problem of crossing number was then found to be NP-complete [2]. However, an algorithm for determining if an embedding was possible and then finding that embedding for fixed $k \geq cr(G)$ was first found in quadratic time [3] and then linear time [5].

There have been algorithms designed for specific spaces, graph families, and embeddings. A $k$-planar embedding is one where the edges of G are split into $k$ graphs: $G = G_1 \cup G_2 \cup \ldots \cup G_k$. The crossing number in this situation is then $cr(G) = \sum cr(G_i)$. Of particular interest is results in 2-planar embeddings [6], [7]. A radial drawing of a bipartite graph is an annulus with the vertices of each part on separate boundaries. This type of embedding has been studied for it's relevance to social networking [8].

# 3 Open Problems to Consider

The biggest open problem to solve would be to combine the linear-time algorithm for embedding a planar graph on an arbetrary fixed surface and one of the polynomial-time algorithms for drawing a graph in $\mathbb{R}^2$ with fewer than $k$ crossings to create a polynomial time algorithm for drawing a graph with fewer than $k$ crossings on an arbetrary fixed surface.

Another open problem would be to study $k$-planar embeddings with the condition that $cr(G_i) = 0$ for $2 \leq i \leq k-1$.

Another open problem would be to change the fixed parameter in the known algorithms. Finding a polynomial time algorithm on a fixed graph to embed with minimum crossings on a surface would be an interesting result. Preliminary results could be found easily for certain family of graphs.

# References


[1] Douglas West: The Art of Combinatorics. (preprint)

[2] Garey, M. R., Johnson, D.S.: Crossing number is NP-complete. SIAM J. Algebra and Discrete Methods, v 4, no. 3 (1984), 312-316

[3] Martin Grohe: Computing crossing numbers in quadratic time. Journal of Computer and System Sciences, v 68, no. 2 (March 2004), 285-302

[4] Bojan Mohar: Embedding graphs in an arbitrary surface in linear time. Proceedings of the twenty-eighth annual ACM symposium on Theory of computing (1996), 392 - 397

[5] Ken-ichi Kawarabayashi, Buce Reed: Computing crossing number in linear time. Proceedings of the thirty-ninth annual ACM symposium on Theory of computing , SESSION: Session 8B, 382 - 390 (2007)

[6] Owens, A. : On the Biplanar Crossing Number. IEEE Trans Circuit Theory CT-18 (1971), 277-280

[7] Éva Czabarka , Ondrej Sýkora , László A. Székely , Imrich Vrto : Biplanar crossing numbers. II. Comparing crossing numbers and biplanar crossing numbers using the probabilistic method. Random Structures & Algorithms , Volume 33 , Issue 4 (December 2008) 480-496

[8] Seok-Hee Hong, Hiroshi Nagamochi: Approximation Algorithms for Minimizing Edge Crossings in Radial Drawings. Algorithmica (2009)