surface
2-manifold
locally X
surface with boundary
torus

CHAPTER 6

Surface Maps

There is nothing "below the surface," my faithful friend—absolutely nothing.

— Letter 52, *The Mahatma Letters to A. P. Sinnett* (1882)

"You have heard of Fortunatus's purse, Miladi? Ah, so! Would you be surprised to hear that, with three of these leetle handkerchiefs, you shall make the Purse of Fortunatus, quite soon, quite easily?"

*"Shall I indeed?" Lady Muriel eagerly replied, as she took a heap of them into her lap, and threaded her needle. "*Please *tell me how, Mein Herr! I'll make one before I touch another drop of tea!"*

— Lewis Carroll, *Sylvie and Bruno Concluded* (1893)

6.1 Surfaces

For most of the rest of the book, we will move from the Euclidean plane to a wider class of topological spaces that *locally* resemble the plane. A ***surface*** or ***2-manifold*** is a topological space[1] that is ***locally homeomorphic*** to the plane; that is, every point in the space lies in an open subset of the space that is homeomorphic to $\mathbb{R}^2$. We also consider ***surfaces with boundary***, spaces in which every point has an open neighborhood homeomorphic to either the plane $\mathbb{R}^2$ or the closed halfspace $\mathbb{R}^2_+ := \{(x, y) \mid x \geq 0\}$.

The simplest examples of surfaces are the plane itself (trivially) and the sphere S^2. For any point x on the sphere, the complement $S^2 \setminus x$ is homeomorphic to the plane via stereographic projection; for any two points x and y, the open sets $S^2 \setminus x$ and $S^2 \setminus y$ cover the sphere.

Another slightly more complex example is the ***torus***. A standard torus is the set of all points at distance r from an arbitrary circle of radius R in $\mathbb{R}^3$, for some real numbers

Klein bottle
Möbius band
surface map
underlying space
abs Sigma
planar map
homeomorphic surface maps

$0 < r < R/2$; the particular choice of parameters is unimportant. In fact, for topological purposes, the shape of the torus is unimportant; the boundary of a coffee cup is also a torus, as is the boundary of a drinking straw. The torus can also be constructed by first gluing two opposite sides of a rectangular sheet of rubber into a cylindrical tube, and then gluing the ends of the tube together into a closed surface. The image of the function $\tau\colon [0,1]^2 \to \mathbb{R}^4$ where $\tau(x,y) = (\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y))$ is a particularly symmetric torus suggested by the gluing construction.

Yet another canonical example is the ***Klein bottle***, first described by Felix Klein in 1882 [47]. The Klein bottle can also be constructed from a sheet of topologist's rubber by gluing opposite pairs of edges, but with one opposite pair twisted 180 degrees. Gluing just the twisted pair together gives us the ***Möbius band***, a one-sided surface with boundary first mentioned in passing by Listing [56,57] and later described in more detail by Möbius [60].[2]

Surfaces arise naturally in many different areas of computing, most visibly in computer graphics [72]. Most surfaces that arise in practice in these areas have some concrete *geometric* representation, usually derived from a collection of polygons in $\mathbb{R}^3$ with explicit vertex coordinates. At least for now, we will ignore all such geometric information; as the joke goes, we do not distinguish between the surface of a donut and the surface of a coffee cup. We do not even require our surfaces to be embedded in $\mathbb{R}^3$ or any other Euclidean space. We also do not require surfaces to have any particular intrinsic geometry; metric notions like distances, areas, angles, and curvatures are (at least for now) neither defined nor relevant.

6.2 Surface Maps

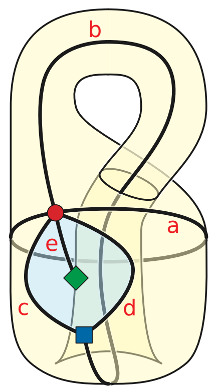
The abstract definition of a 2-manifold is difficult to leverage for actual computation. Instead, almost all algorithms (and most proofs!) for surfaces rely on a decomposition of surfaces into topologically simple pieces. A ***map***[3] on a compact connected surface $\mathcal{S}$ is a triple $\Sigma = (V, E, F)$ with the following components.

- V is a non-empty finite set of points in $\mathcal{S}$, called *vertices*.
- E is a finite set of vertex-to-vertex paths in $\mathcal{S}$, called *edges*, which are simple and disjoint except possibly at their endpoints. Thus, V and E are the vertices and edges of a topological graph G.
- F is the set of components of $\mathcal{S} \setminus G$, called *faces*, each of which is homeomorphic to an open disk. As we will see later, this constraint implies that the graph G is connected.

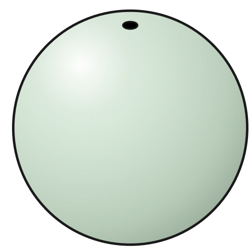
We call the 2-manifold $\mathcal{S}$ the ***underlying space*** of the map Σ, denoted $|\Sigma|$. A map is ***planar*** if its underlying surface is the sphere. Two maps Σ and Σ' on the same surface $\mathcal{S}$ are ***homeomorphic*** if there is a continuous function $h\colon \mathcal{S} \to \mathcal{S}$ that sends each vertex, edge, and face of Σ homeomorphically to a distinct vertex, edge, or face of Σ', respectively. Homeomorphic surface maps are usually considered to be identical.

Recall that a loop in a graph is an edge that is incident to only one vertex, and a bridge is an edge whose deletion disconnects the graph. An ***isthmus*** in a surface map is an edge that is incident to only one face. As in planar maps, every bridge in a surface map is also an isthmus, but an isthmus in a surface map is not necessarily a bridge. Moreover, unlike in planar graphs, the same edge in a surface map can be both a loop and an isthmus; the Jordan curve theorem does not extend to more general surfaces.

loop
bridge
isthmus
trivial map
exercises for the reader
polygonal schema
faces
corner
side
direction
label



A map on the Klein bottle. Edges a, b, and e are isthmuses; edge a is a loop; and edge e is a bridge.



The trivial map.

Most of the exposition in this chapter (and in later chapters) implicitly assumes that the underlying surface map has at least one edge. Exactly one surface map violates this assumption, namely the ***trivial map*** of the sphere, which has one vertex and one face. All of the results in this chapter apply (trivially!) to the trivial map; however, some proofs require minor modifications, which are left as exercises for the reader.

In the rest of this section, we describe several different combinatorial representations of surface maps, as instructions for gluing polygons together along edges, as a description of an embedding of a graph onto the surface, or as a set of three involutions.

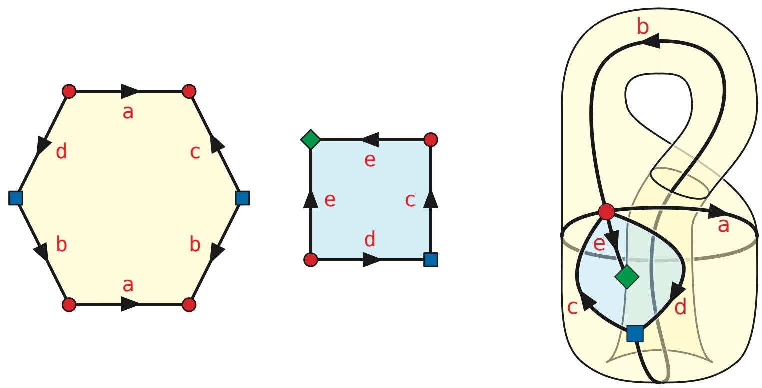
Polygonal Schemata (Sides)

Informally, a ***polygonal schema*** is a finite collection of polygons with rules for gluing them together into a surface. The constituent polygons are called the ***faces*** of the schema; to avoid later confusion, the vertices of each face are called ***corners***, and the edges of each faces are called ***sides***. Each side of each face is assigned a ***direction*** and a ***label*** from some finite set, so that each label is assigned exactly twice. Then we can construct a surface from a polygonal schema by identifying each pair of sides with matching labels so that their directions coincide. For example, the torus and the Klein bottle can both be obtained by identifying opposite sides of a square, but with differently directed sides.

We can encode any polygonal schema by listing the labels and orientations of sides in cyclic order around each face. Whenever we traverse a side along its chosen direction,

signature
signed polygonal schema
Surface Pi
edge
vertex
Sigma Pi

we record its label; when we traverse an edge against its chosen direction, we record its label with a bar over it. Thus, in the example schema shown below, traversing each polygon clockwise starting from its upper left corner, we obtain the signature $(\mathsf{a}\bar{\mathsf{c}}\mathsf{b}\bar{\mathsf{a}}\bar{\mathsf{b}}\bar{\mathsf{d}})(\bar{\mathsf{e}}\bar{\mathsf{c}}\bar{\mathsf{d}}\mathsf{e})$. The signature of a polygonal schema is not unique—we could traverse the faces in a different order, start at a different side of each face, or traverse some faces in the opposite direction. Thus, $(\bar{\mathsf{d}}\mathsf{e}\bar{\mathsf{e}}\bar{\mathsf{c}})(\bar{\mathsf{a}}\mathsf{d}\mathsf{b}\mathsf{a}\bar{\mathsf{b}}\mathsf{c})$ and $(\bar{\mathsf{c}}\mathsf{b}\bar{\mathsf{a}}\bar{\mathsf{b}}\bar{\mathsf{d}}\mathsf{a})(\mathsf{c}\mathsf{e}\bar{\mathsf{e}}\mathsf{d})$ are also valid signatures for the example schema below.



A polygonal schema with signature $(\mathsf{a}\bar{\mathsf{c}}\mathsf{b}\bar{\mathsf{a}}\bar{\mathsf{b}}\bar{\mathsf{d}})(\bar{\mathsf{e}}\bar{\mathsf{c}}\bar{\mathsf{d}}\mathsf{e})$; arrows indicate edge directions.

More formally, an ***signed polygonal schema*** is a tuple $\Pi = (S, \mathit{next}, \mathit{flip}, \mathit{fsign})$ with the following components:

- S is a finite set of abstract objects called *sides*.
- $\mathit{next}\colon S \to S$ is a permutation of S, whose orbits are called *faces*.
- $\mathit{flip}\colon S \to S$ is an involution of S, whose orbits are called *edges*.
- $\mathit{fsign}\colon S \to \{-1, +1\}$ is a function such that $\mathit{fsign}(s) = \mathit{fsign}(\mathit{flip}(s))$.

For any polygonal schema Π, we define a corresponding topological space $\mathbf{\mathcal{S}(\Pi)}$ as follows. Let F denote a set of disjoint closed disks in the plane, one for each orbit of *next*. For each face of degree d, we subdivide the boundary of the corresponding disk into d paths, directed clockwise around the disk. We identify these paths with the sides S so that $\mathit{next}(s)(0) = s(1)$ for every side $s \in S$. Then the space $\mathcal{S}(\Pi)$ is the quotient space $\bigsqcup F/{\sim}$, where for all sides $s \in S$ and all $t \in [0,1]$, we define

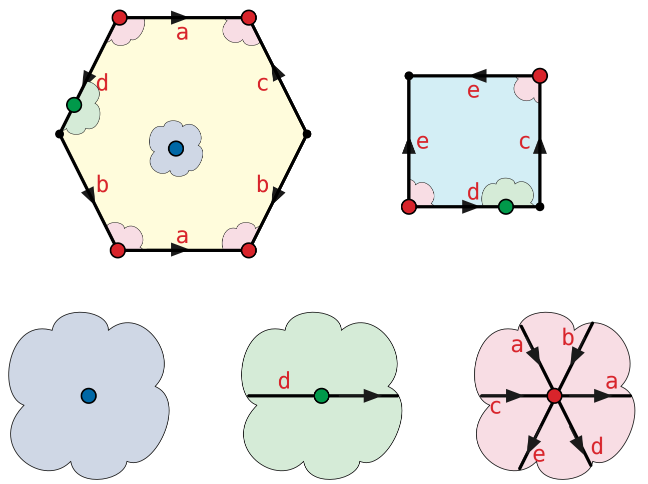
$$s(t) \sim \begin{cases} \mathit{flip}(s)(t) & \text{if } \mathit{fsign}(s) = -1, \\ \mathit{flip}(s)(1-t) & \text{if } \mathit{fsign}(s) = +1. \end{cases}$$

Thus, the function *fsign* indicates whether any side and its flip should be identified with the same or opposite directions. Each pair of identified sides in Π becomes a single undirected path in the surface $\mathcal{S}(\Pi)$, which we call an ***edge***. The identification of sides induces an identification of corners to a smaller number of points on the surface, which we call ***vertices***. The vertices, edges, and faces constitute a map $\mathbf{\Sigma(\Pi)}$ of the space $\mathcal{S}(\Pi)$.

trivial map
nets
embedding
face
cellular embedding

(This construction breaks down when $S = \varnothing$; in this case, $\mathcal{S}(\Pi)$ is the sphere and $\Sigma(\Pi)$ is the trivial map.)

It is not hard to show that the surface $\mathcal{S}(\Pi)$ is always a compact 2-manifold. Any point in the interior of a face has an open disk neighborhood *by definition*. Any point in the interior of an edge e of $\Sigma(\Pi)$ has neighborhoods that intersect only the faces on either side of e; the union of the interiors of those faces and the interior of e is an open disk. Finally, consider the graph whose nodes are the corners of F and whose edges join corners that share an edge in Σ. Every corner has exactly two neighbors, so this graph is the union of disjoint cycles; by definition, the corners in each cycle define a vertex of Σ. It follows that any vertex v has a neighborhood homeomorphic to an open disk subdivided into wedges by its incident edges.



Neighborhoods of three points in the example schema.

Polygonal schemata originate with nets of convex polyhedra, first used by Albrecht Dürer in the early 16th century [22, 23]. Although they are implicit in the earlier work of Riemann [68] and Jordan [45], polygonal schemata for more general surfaces, both orientable and non-orientable, were first explicitly described by Möbius [60]. Polygonal schemata were firmly established as canonical representations of Riemannian surfaces by Dyck [24] and Klein [48].[4]

Rotation Systems (Darts)

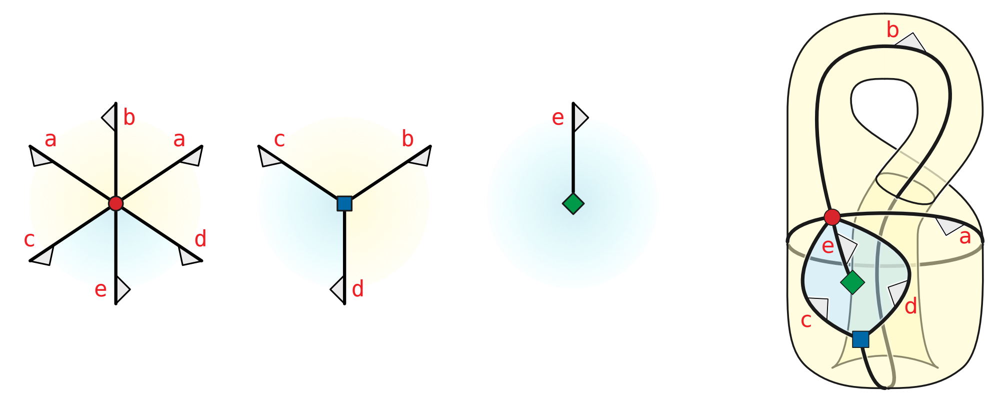
The definition of a planar or spherical graph embedding generalizes naturally to arbitrary surfaces: An embedding of a (topological) graph G into a surface Σ is a continuous injective map from G into Σ. As in the planar setting, we often use the same symbol G to denote an abstract graph, the corresponding topological graph, and the image of an embedding of that graph. For any embedded graph G in any surface Σ, the components of $\Sigma \setminus G$ are called the ***faces*** of the embedding. An embedding is ***cellular*** if every face is

orientation of an edge
left shore
signature
signed rotation system

homeomorphic to an open disk, or equivalently, if the vertices, edges, and faces of the embedding define a surface map.

Any graph embedding defines a cyclic ordering on the darts leaving any vertex. Suppose we assign each dart an ***orientation***, designating one of its incident faces as its left shore, and a label from some finite set, so that each label is assigned exactly twice. (The word “left” is just a convenient label; it carries no geometric meaning.) The labels indicate which pairs of darts constitute the edges of the graph, and the orientations describe the neighborhood of each edge in the target surface.

We can encode any *cellular* embedding by listing the labels and orientations of the darts in cyclic order around each vertex. Whenever we cross a dart from left to right (according to its assigned orientation), we record its label; whenever we cross a dart from right to left, we record its label with a bar over it. Thus, in the example embedding shown below, if we walk around each vertex counterclockwise starting with the lowest dart, we obtain the signature $(\mathsf{e}\bar{\mathsf{d}}\bar{\mathsf{a}}\mathsf{bac})(\bar{\mathsf{d}}\mathsf{bc})(\bar{\mathsf{e}})$. Just as for polygonal schemata, the signature of a cellular embedding is not unique.



A signed rotation system with signature $(\mathsf{e}\bar{\mathsf{d}}\bar{\mathsf{a}}\mathsf{bac})(\bar{\mathsf{d}}\mathsf{bc})(\bar{\mathsf{e}})$; triangles indicate edge orientations.

More formally, a ***signed rotation system*** is a quadruple $(D, rev, succ, rsign)$ with the following components.

- D is a finite set of abstract objects called *darts*.
- $rev : D \to D$ is an involution, whose orbits are called *edges*.
- $succ : D \to D$ is a permutation, whose orbits are called *vertices*.
- $rsign : D \to \{-1, +1\}$ is a function such that $rsign(d) = rsign(rev(d))$ for every dart d.

The function *rsign* records whether the cyclic orders at the endpoints of each edge are consistent (↺—↺ or ↻—↻) or inconsistent (↺—↻ or ↻—↺).

The study of surface graph embeddings was instigated by Heawood [39], who asked for the minimum number of colors required to properly color a map on an arbitrary orientable surface. Heffter [40] posed the dual problem of coloring the vertices of a surface-embedded graph; he also used polygonal schemata to encode cellular embeddings on orientable surfaces; Ringel [69] extended this representation to non-orientable surfaces. Brahana and Coble [6, 7] essentially used rotation systems to represent *regular* surface maps.[5] (Unsigned) rotation systems for arbitrary simple graphs were introduced by Edmonds [25] (apparently independently from earlier work) and formalized by Youngs [89]; Youngs also introduced signed rotation systems for simple graphs on non-orientable surfaces [90]. Similar representations were independently proposed by Jacques [42, 43], Biggs [2], and Walsh and Lehman [88]. Tutte [81] and Gross and Alpert [32] extended both unsigned and signed rotation systems to general graphs; Gross and Alpert called the cellular embedding induced by a signed rotation system a *Heffter-Edmonds imbedding*. Graphs equipped with rotation systems are also called *vortex graphs* [89], *constellations* [43], *circularized 1-complexes* [50, 51], *combinatorial maps* [44, 81], *fat graphs* [63], *cyclic graphs* [4], *ribbon graphs* [5], and (pre-clean) *dessins d'enfants* [73]. Following Jacques [42, 43], some authors refer to darts in surface maps as *brins*, from the French word for "strand" [15].

regular
Heffter-Edmonds imbedding
vortex graphs
constellations
circularized 1-complexes
combinatorial maps
fat graphs
cyclic graphs
ribbon graphs
dessins d'enfants
brins
Band Sigma
band decomposition
trivial map
end!of a band
side!of a band

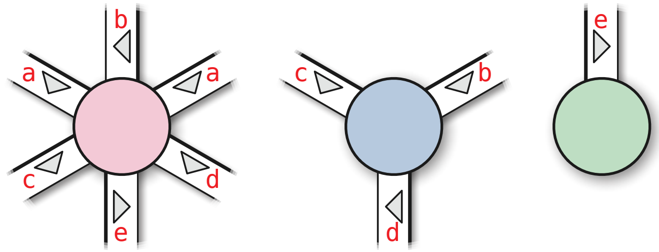
Band Decompositions

Just like polygonal schemata, each signed rotation system describes a unique surface map (up to homeomorphism, of course). Instead of describing this map Σ directly, it is convenient to first consider a related surface map $\Sigma^\square$, called the ***band decomposition*** of Σ [33], intuitively obtained by shrinking each face of Σ slightly, expanding each vertex of Σ into a small closed disk, and replacing each edge of Σ with a closed quadrilateral "band". (The band decomposition of the trivial map consists of two closed disks glued along their boundaries; formally, this decomposition of the sphere is not a map, because it has no vertices!)

To construct the band decomposition $\Sigma(\Pi)^\square$ from a signed rotation system Π, we associate a closed disk $v^\square$ with each vertex v and a closed quadrilateral $e^\square$ with each edge e. Two opposite edges of every band are called its ***ends***; the other two edges are called its ***sides***. The ends of each band $e^\square$ are attached to the boundaries of the disks corresponding to the endpoints of e. The attached bands meet each vertex disk in a sequence of disjoint closed paths, in the counterclockwise order specified by the permutation *succ*. Bands corresponding to positive edges ($rsign = +1$) are attached without twisting; bands corresponding to negative edges ($rsign = -1$) are attached with a half-twist. The union of the vertex disks and edge bands is a surface with boundary; gluing a disk $f^\square$ to each boundary cycle f completes the band decomposition.

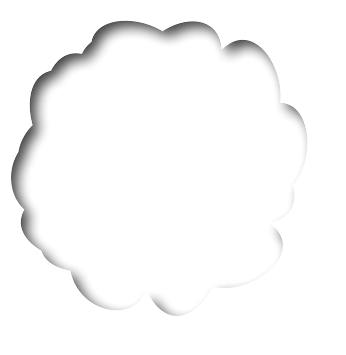
Band decompositions can be constructed from any signed polygonal schema Π in a similar way. To each face f in the schema, we associate a disk $f^\square$ with twice as many sides as f. For each edge e, we attach the *sides* of a band $e^\square$ to the disks corresponding to its incident faces, untwisted if $fsign(e) = +1$ or twisted if $fsign(e) = -1$. The bands are attached to alternate sides of each disk $f^\square$ in the clockwise order specified by the permutation *next*. Finally, to each boundary cycle of the surface constructed so far, we

one can mechanically verify
One can mechanically verify
blade
flag



Expanding a signed rotation system into a band decomposition.

Untwisted and twisted bands.



The band decomposition of the example Klein bottle map.

attach a disk $v^\square$; one can mechanically verify that these disks correspond to the vertices of the surface map $\Sigma(\Pi)$.

Given a band decomposition $\Sigma^\square$, we can recover the original surface map Σ by placing a vertex v in the interior of each disk $v^\square$ and threading a vertex-to-vertex path e through each band $e^\square$. One can mechanically verify that each face f of the resulting map contains exactly one face $f^\square$ of the band decomposition.

By construction, the band decomposition $\Sigma^\square$ has three types of faces, corresponding to the vertices, edges, and faces of the original map Σ; adjacent faces in $\Sigma^\square$ are always of different types. Similarly, $\Sigma^\square$ has three types of edges, correspond to the sides, corners, and darts of Σ. Finally, each vertex of $\Sigma^\square$ corresponds to a ***blade*** in Σ: an edge with a direction, specifying which of its endpoints is the tail, and an *independent* orientation, specifying which of its shores is on the left. Blades are also commonly called ***flags***. Each vertex of $\Sigma^\square$ is incident to exactly one edge and one face of each type.

reflection system
vertices
edges
faces
trivial map

The construction of band decompositions from rotation systems was proposed independently by Ladegaillerie [50,51] and Lins [54,55]; however, band decompositions themselves are already implicit in earlier works [19,31,80].

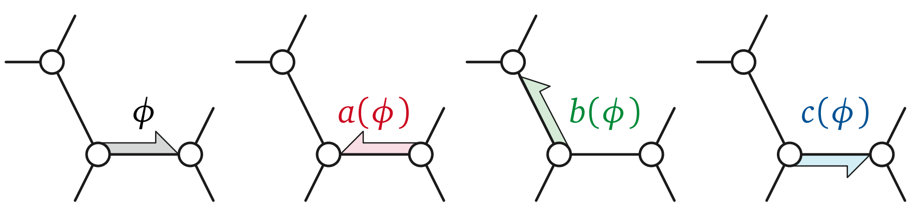
Reflection Systems (Blades)

The structure of the band decomposition motivates the following abstract representation of surface maps. An ***reflection system*** is a quadruple $\Xi = (\Phi, a, b, c)$ where

- Φ is a finite set of abstract objects called *blades*;
- $a\colon \Phi \to \Phi$ is an involution of Φ, whose orbits are called *sides*;
- $b\colon \Phi \to \Phi$ is an involution of Φ, whose orbits are called *corners*;
- $c\colon \Phi \to \Phi$ is an involution of Φ, whose orbits are called *darts*; and
- $ac = ca$ is also an involution of Φ.

To simplify notation, we will use concatenation to denote compositions of the involutions a, b, and c; thus, for example, ac is shorthand for the permutation $a \circ c$, and $abc(\phi)$ is shorthand for $a(b(c(\phi)))$.

Recall that each blade in a map Σ can be associated with a triple (v, e, f), where e is an edge of Σ, v is one of the endpoints of e, and f is one of the shores of e. Intuitively, each of the three involutions in Ξ changes one of the components of this triple: a changes the vertex (or **a**pex); b changes the edge (or **b**order), and c changes the face (or **c**hamber). More formally, the vertices of the reflection system are the orbits of the permutation group $\langle b, c\rangle$; the orbits of $\langle a, c\rangle$ are its edges; and the orbits of $\langle a, b\rangle$ are its faces. By definition, every vertex and face contains an even number of blades, and every edge contains exactly four blades.

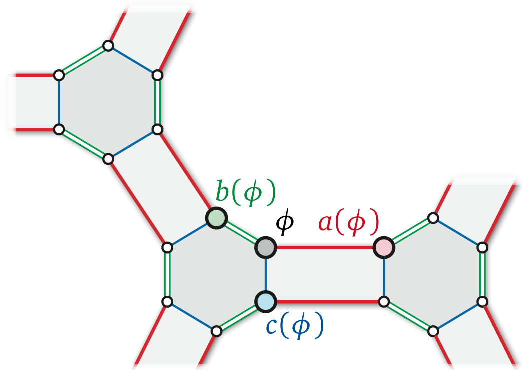


Operations on blades in a reflection system.

Equivalently, each blade in a reflection system for a map Σ corresponds to a vertex of the band decomposition $\Sigma^\square$, and each of the involutions a, b, c correspond to one of the three types of edges of $\Sigma^\square$.

Any reflection system represents a unique surface map with corresponding blades, sides, corners, darts, vertices, edges, and faces. (In particular, the unique reflection system with $\Phi = \varnothing$ represents the trivial map.) Conversely, unlike signed polygonal schemata and signed rotation systems, every surface map is represented by a *unique* reflection system.

combinatorial map
map (reflection system)



Corresponding vertices in the band decomposition.

Reflection systems are more commonly called ***combinatorial maps*** [14, 84] or even just ***maps*** [81, 87], at least in contexts where they will not be confused with the topological structures they encode (or with continuous functions).[6] Reflection systems were first proposed by Coxeter and Moser [17] as a representation of *regular* maps, building on Coxeter's earlier seminal work on reflection groups [16], which in turn was motivated by Hamilton's Icosian calculus [36, 37, 38]. Their use as a representation for *arbitrary* maps was proposed independently by many different authors, including Robertson [70], Tutte [81], Cooper [14], Walsh [87], Ferri [28, 29], Voisin and Malgoire (building on earlier ideas of Grothendieck) [85, 86], Lins [54, 55], and Vince [84].

Equivalence

Given a signed polygonal schema $(S, \mathit{next}, \mathit{flip}, \mathit{fsign})$, a reflection system (Φ, a, b, c) for the same map can be defined as follows:

$$\begin{aligned}
\Phi &:= S \times \{-1, +1\} \\
a(s, \varepsilon) &:= (\mathit{flip}(s), -\varepsilon \cdot \mathit{fsign}(s)) \\
b(s, \varepsilon) &:= (\mathit{next}^{\varepsilon}(s), -\varepsilon) \\
c(s, \varepsilon) &:= (s, -\varepsilon)
\end{aligned} \tag{6.1}$$

(Here, $\mathit{next}^{\varepsilon}(s)$ is shorthand for $\mathit{next}(s)$ if $\varepsilon = +1$ or $\mathit{next}^{-1}(s)$ if $\varepsilon = -1$.) Similarly, given a signed rotation system $(D, \mathit{succ}, \mathit{rev}, \mathit{rsign})$, the reflection system (Φ, a, b, c) for the same map can be defined as follows:

$$\begin{aligned}
\Phi &:= D \times \{-1, +1\} \\
a(d, \varepsilon) &:= (d, -\varepsilon) \\
b(d, \varepsilon) &:= (\mathit{succ}^{\varepsilon}(d), -\varepsilon) \\
c(d, \varepsilon) &:= (\mathit{rev}(d), -\varepsilon \cdot \mathit{rsign}(d))
\end{aligned} \tag{6.2}$$

dual map

Conversely, suppose $\Xi = (\Phi, a, b, c)$ is the reflection system of some surface map Σ. Recall that a face of Ξ is an orbit of the permutation group $\langle a, b\rangle$. Suppose we label the blades in each face cycle alternately *positive* and *negative*. (For each face, there are exactly two ways to label its blades; choose one arbitrarily.) Let Φ^+ and Φ^- respectively denote the subsets of positive and negative blades. Now we can define an oriented polygonal schema $(S, \mathit{next}, \mathit{flip}, \mathit{fsign})$ for Σ as follows:

$$
\begin{aligned}
S &:= \Phi^+\\
\mathit{next}(\phi^+) &:= ab(\phi^+)\\
\mathit{flip}(\phi^+) &:= \begin{cases} c(\phi^+) & \text{if } c(\phi^+) \in \Phi^+\\ ac(\phi^+) & \text{if } c(\phi^+) \in \Phi^- \end{cases}\\
\mathit{fsign}(\phi^+) &:= \begin{cases} -1 & \text{if } c(\phi^+) \in \Phi^+\\ +1 & \text{if } c(\phi^+) \in \Phi^- \end{cases}
\end{aligned}
\tag{6.3}
$$

Similarly, a vertex of Ξ is an orbit of the permutation group $\langle b, c\rangle$. Label the blades in each *vertex* cycle alternately positive and negative, and let Φ_+ and Φ_- respectively denote the subsets of positive and negative blades. Then an oriented rotation system $(D, \mathit{succ}, \mathit{rev}, \mathit{rsign})$ for Σ can be defined as follows:

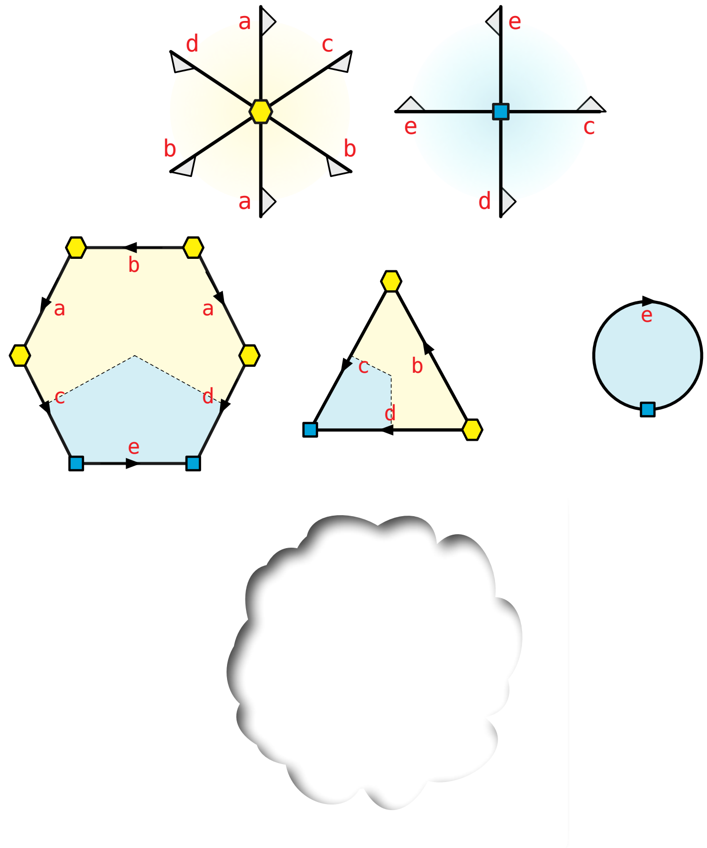
$$
\begin{aligned}
D &:= \Phi_+\\
\mathit{succ}(\phi_+) &:= cb(\phi_+)\\
\mathit{rev}(\phi_+) &:= \begin{cases} a(\phi_+) & \text{if } a(\phi_+) \in \Phi_+\\ ca(\phi_+) & \text{if } a(\phi_+) \in \Phi_- \end{cases}\\
\mathit{rsign}(\phi_+) &:= \begin{cases} -1 & \text{if } a(\phi_+) \in \Phi_+\\ +1 & \text{if } a(\phi_+) \in \Phi_- \end{cases}
\end{aligned}
\tag{6.4}
$$

In conclusion, any of our three abstract representations for a surface map can be converted into any other in $O(m)$ time, where m is the number of edges in the map.

6.3 Duality

The syntactic similarity between polygonal schemata and rotation systems is not coincidental; the two structures are in fact dual to each other. Any signed polygonal schema for any map Σ *is* also a signed rotation system for another surface map Σ^*, called the ***dual map*** of Σ. Each side s, edge e, face f, vertex v, dart d, corner c, or blade ϕ of Σ corresponds to—or more evocatively, "is dual to"—a distinct dart s^*, edge e^*, vertex f^*, face v^*, side d^*, corner c^*, or blade ϕ^* of the dual map Σ^*, respectively. Symmetrically, any signed rotation system for any map Σ is also a signed polygonal schema for the same dual map Σ^*.

exercise for the reader



The dual of our example map on the Klein bottle.

These correspondences follow directly from the following observation, whose easy proof we leave as an exercise for the reader:

Lemma 6.1. *Let Π be a signed polygonal schema for a surface map Σ. The band decomposition $\Sigma^\square$ constructed from Π is **identical** to the band decomposition $(\Sigma^*)^\square$ constructed from Π as a signed rotation system.*

This lemma has several other immediate consequences:

Corollary 6.2. *Dual maps Σ and Σ^* have identical band decompositions: $(\Sigma^*)^\square = \Sigma^\square$.*

Corollary 6.3. *Dual maps Σ and Σ^* lie on the same underlying surface.*

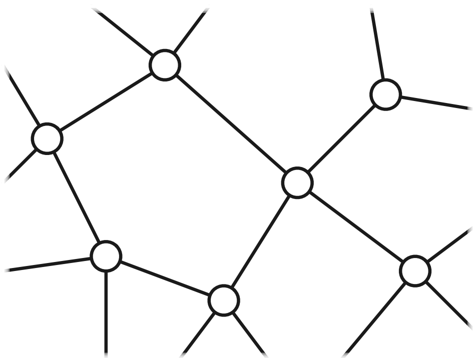
Corollary 6.4. *Each surface map Σ has a unique dual map Σ^* (up to homeomorphism).*

Corollary 6.5. *Surface map duality is an involution: $(\Sigma^*)^* = \Sigma$.*

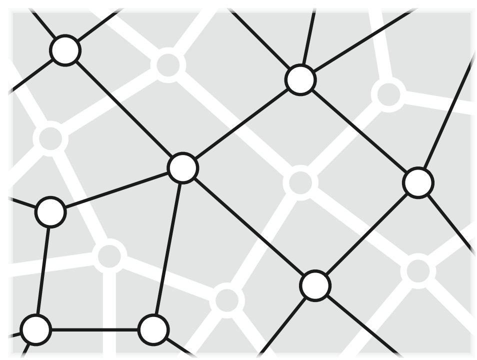
dual map
One can verify mechanically

Similarly, if $\Xi = (\Phi, a, b, c)$ is the reflection system for Σ, exchanging the involutions a and c gives us the dual reflection system $\Xi^* = (\Phi, c, b, a)$ for Σ^*. (As a mnemonic device, we can refer to vertices, edges, and faces of Σ^* as **a**reas, **b**orders, and **c**entroids, respectively.)

Just as in the planar setting, dual maps can also be defined directly in terms of their topology, as follows. Two maps Σ and Σ^* of the same underlying surface are ***duals*** if and only if each face of Σ contains exactly one vertex of Σ^*, each face of Σ^* contains exactly one vertex of Σ, and each edge of Σ crosses exactly one edge of Σ^*. One can verify mechanically that if two maps Σ and Σ^* satisfy these three conditions, then any signed polygonal schema for Σ is a signed rotation system for Σ^* and vice versa.



A region of a surface map Σ



The corresponding region of the dual map Σ^*

Unlike the planar setting, however, duality does *not* imply a correspondence between the darts of Σ and the *darts* of Σ^*. The table on the next page summarizes several important correspondences between surface maps and their duals.

primal Σ	**dual Σ^***
vertex v	face v^*
edge e	edge e^*
face f	vertex f^*
blade ϕ	blade ϕ^*
dart d	side d^*
corner c	corner c^*
side s	dart s^*
tail(d)	*left*(d^*)
head(d)	*right*(d^*)
left(s)	*tail*(s^*)
right(s)	*head*(s^*)

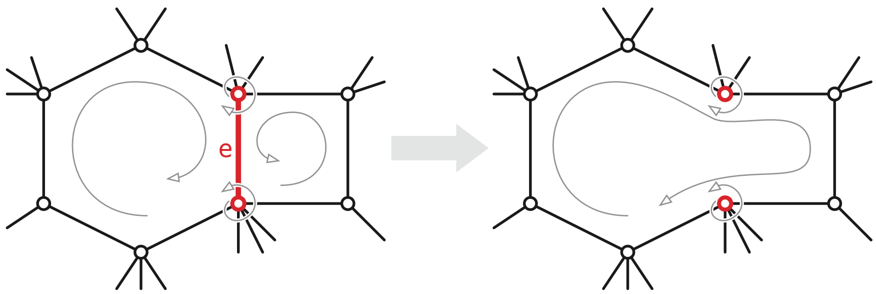
primal Σ	**dual Σ^***
underlying surface $\lvert\Sigma\rvert$	underlying surface $\lvert\Sigma\rvert$
polygonal schema	rotation system
reflection system (Φ, a, b, c)	reflection system (Φ, c, b, a)
band decomposition $\Sigma^\square$	band decomposition $\Sigma^\square$
medial map $\Sigma^\times$	radial map $\Sigma^\diamond$
boundary loop	bridge
loop	isthmus
cycle	cocycle
boundary subgraph	edge cut
deletion $\Sigma \setminus e$	contraction Σ^* / e^*
contraction Σ / e	deletion $\Sigma^* \setminus e^*$

Correspondences between features of primal and dual surface maps

delete an edge in a surface map
Sigma without e
reflect a face of a polygonal schema
contract an edge in a surface map
Sigma contract e
reflect a vertex of a rotation system

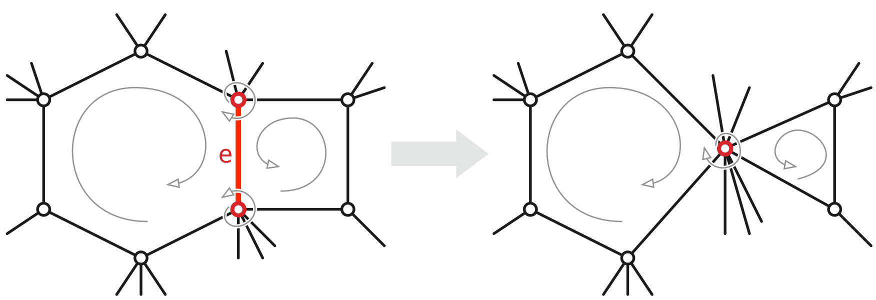
Deletion and Contraction

Let e be an edge in a surface map Σ. If e is not an isthmus, we can ***delete*** e to obtain a new map $\boldsymbol{\Sigma \setminus e}$ on the same underlying surface. The two shores of e are merged into a single face in $\Sigma \setminus e$. If Σ is represented by a signed rotation system, we simply delete both darts of e. If Σ is represented by a signed polygonal schema, however, we must merge the cyclic orders of sides in the incident faces. Moreover, if $\mathit{fsign}(e) = -1$, we first ***reflect*** one of the two faces, reversing the cyclic order of its sides and changing the signs of its sides and their flips. (Reflecting the only face incident to an isthmus changes the sign of that isthmus twice, and therefore not at all.)



Deleting an edge between differently oriented faces.

Symmetrically, if e is not a loop, we can ***contract*** e to obtain a new map $\boldsymbol{\Sigma / e}$ on the same underlying surface. The two endpoints of e are merged into a single vertex in Σ / e. If Σ is represented by a signed polygonal schema, we simply delete both sides of e. If Σ is represented by a signed polygonal schema, however, we must merge the cyclic orders of darts leaving the incident vertices. Moreover, if $\mathit{rsign}(e) = -1$, we first ***reflect*** one of the two vertices, reversing the cyclic order of its darts and changing the signs of its darts and their reversals. (Reflecting the single endpoint of a loop changes the sign of that loop twice, and therefore not at all.)



Contracting an edge between differently oriented vertices.

Finally, if the map Σ is represented by an reflection system $\Xi = (\Phi, a, b, c)$, both contraction and deletion can be implemented by removing four blades and changing the involution b. If we equate e with the corresponding set of four blades in Φ, then the

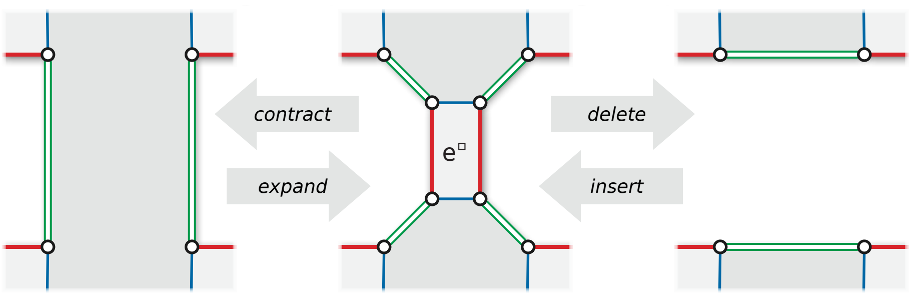
edge insertion
edge expansion

reflection systems $\Xi \setminus e = (\Phi \setminus e, a, b \setminus e, c)$ and $\Xi / e = (\Phi \setminus e, a, b / e, c)$, which respectively represent the maps $\Sigma \setminus e$ and Σ / e, can be defined as follows, for all $\phi \in \Phi \setminus e$:

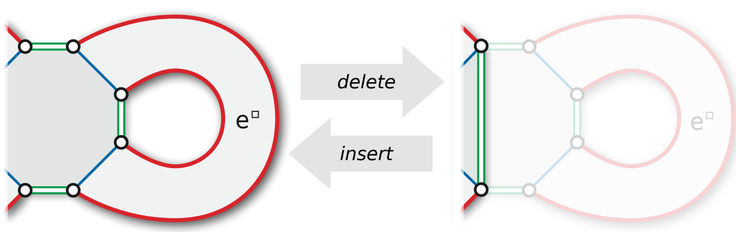
$$(b \setminus e)(\phi) := \begin{cases} bcbcb(\phi) & \text{if } b(\phi) \in e \text{ and } bcb(\phi) \in e \\ bcb(\phi) & \text{if } b(\phi) \in e \\ b(\phi) & \text{otherwise} \end{cases} \tag{6.5}$$

$$(b / e)(\phi) := \begin{cases} babab(\phi) & \text{if } b(\phi) \in e \text{ and } bab(\phi) \in e \\ bab(\phi) & \text{if } b(\phi) \in e \\ b(\phi) & \text{otherwise} \end{cases} \tag{6.6}$$

The complicated first case of $b \setminus e$ occurs when e is the only edge incident to some face; symmetrically, the first case of b / e occurs when e is the only edge incident to some vertex. We emphasize the involutions a and c from the original reflection system Ξ appear verbatim (except for their smaller domains) in $\Xi \setminus e$ and Ξ / e.



Contraction, expansion, deletion, and insertion in the band decomposition.



Deleting or inserting a loop that bounds a face.

Lemma 6.6. *Let e be an arbitrary edge in an arbitrary surface map Σ. If e is not an isthmus, then $\Sigma \setminus e$ and Σ lie on the same underlying surface. Symmetrically, if e is not a loop, then Σ / e and Σ lie on the same underlying surface.*

The inverse of edge deletion is called ***insertion***, and the inverse of edge contraction is called ***expansion***. To fully specify an edge insertion, we must specify not only the

Sigma minus X mod Y
system of loops
reduced polygonal schema
reduced rotation system
face cut
boundary
boundary subgraph
cocycle

endpoints of the new edge, but the face f to be split by the new edge and the corners of f that the new edge will connect. Similarly, to fully specify an edge expansion, we must specify not only the faces on either side of the new edge, but the vertex v to be split by the new edge and the partition of edges incident to v.

Deletion and contraction are clearly dual operations, just as they are in planar maps.

Lemma 6.7 (Contraction $\rightleftharpoons$ deletion). *Let e be an arbitrary edge in an arbitrary surface map Σ. If e is not a loop, then $(\Sigma / e)^* = \Sigma^* \setminus e^*$. Symmetrically, if e is not an isthmus, then $(\Sigma \setminus e)^* = \Sigma^* / e^*$.*

For any disjoint subsets X and Y of the edges of Σ, we write $\boldsymbol{\Sigma / X \setminus Y}$ to denote the map resulting from contracting every edge in X and deleting every edge in Y. The order of contractions and deletions is unimportant; any permutation of the operations leads to the same map. If X contains no cycles and Y contains no cocycles, $\Sigma / X \setminus Y$ lies on the same underlying surface as Σ.

A ***system of loops*** is a surface map with exactly one vertex and exactly one face. Every edge in a system of loops is both a loop and an isthmus; thus, the dual of a system of loops is another system of loops. The trivial map is (trivially) a system of zero loops. Any system of loops can be represented by a ***reduced polygonal schema***, which is a signed polygonal schema with one face, whose corners all identify to a single vertex. The dual of a reduced polygonal schema is a ***reduced rotation system***.

Cocycles and Boundary Subgraphs

Unlike planar maps, the dual of a cycle in a surface map is not necessarily a bond, and the dual of a bond is not necessary a cycle, so we must introduce some new terminology.

A ***face cut*** in a surface map Σ is a partition of its faces F into two non-empty subsets R and $F \setminus R$. The ***boundary*** of a face cut $(R, F \setminus R)$ consists of all edges with one shore in X and the other shore in $F \setminus R$. A ***boundary subgraph*** is the boundary of some facet cut. Equivalently, X is a boundary subgraph of Σ if and only if X^* is an edge cut in Σ^*.

Lemma 6.8. *Every boundary subgraph of a surface map Σ is an even subgraph of Σ.*

Proof: Let X be the boundary of a face cut $(R, F \setminus R)$ in some surface map Σ. By definition, X^* is the boundary of the vertex cut $(R^*, F^* \setminus R^*)$ in Σ^*. Any closed walk in Σ^* that starts at a vertex of R must also end in a vertex of R. Thus, every closed walk in Σ^* contains an even number of edges in X^*. In particular, every face of Σ^* is incident to an even number of edges of X^*. Equivalently, every vertex of Σ is incident to an even number of edges of X. □

A ***cocycle*** in a map Σ is a subgraph C such that corresponding dual subgraph $C^* = \{e^* \mid e \in C\}$ is a cycle in Σ^*. Euler's theorem [27] implies that every even subgraph

spanning tree
spanning cotree
tree-cotree decomposition
leftover
oriented polygonal schema

(and thus every boundary subgraph) of a surface map is the union of edge-disjoint cycles; thus, every edge cut in a surface map is the union of edge-disjoint cocycles.

Tree-Cotree Decompositions

The duality between contraction and deletion immediately implies the following weak generalization of Corollary **??**. Recall that a spanning tree of a graph is a connected acyclic subgraph that includes every vertex; a spanning tree of a map Σ is just a spanning tree of the vertices and edges of Σ. If T is a spanning tree of Σ, the corresponding dual subgraph $T^* = \{e^* \mid e \in T\}$ is a called ***spanning cotree*** of the dual map Σ^*.

Lemma 6.9 (Biggs [3]). *Let T be an arbitrary spanning tree of an arbitrary surface map Σ. The map $(\Sigma / T)^* = \Sigma^* \setminus T^*$ contains a spanning tree of Σ^*.*

Proof: For any edge e in T, the corresponding dual edge e^* is not a bridge in Σ^*. Thus, the edges of $\Sigma^* \setminus T^*$ constitute a connected subgraph of Σ^*. □

A ***tree-cotree decomposition*** of a surface map $\Sigma = (V, E, F)$ is a partition (T, L, C) of its edges into three disjoint subgraphs: a spanning tree T, a spanning cotree C, and ***leftover*** edges $L = E \setminus (T \cup C)$. Lemma 6.9 implies that any spanning tree T is part of some tree-cotree decomposition, as is any spanning cotree C. Corollary **??** implies that if (T, L, C) is a tree-cotree decomposition of a *planar* map, then $L = \varnothing$; in the next chapter, we will prove that $L = \varnothing$ if *and only if* the decomposed map is planar. If (T, L, C) is a tree-cotree decomposition of a surface map Σ, then $\Sigma / T \setminus C$ is a system of loops, in which each edge in L survives as a loop.

Lemma 6.10 (Eppstein [26]). *Let Σ be a surface map with distinct edge weights. The minimum spanning tree of Σ and the maximum spanning cotree of Σ are disjoint.*

Proof: Let e be an edge in the minimum spanning tree T, and let X be the subset of edges of Σ with one endpoint in each component of $T \setminus e$. Lemma **??** implies that e is the lightest edge in the bond X. So Lemma 6.8 implies that e^* is the minimum weight edge in the even subgraph X^* and thus is also the minimum-weight edge in some cycle in Σ^*. We conclude that e^* is not an edge of the maximum spanning tree of Σ^*. □

6.4 Orientability

To construct a signed polygonal schema for a given surface map, we must choose an arbitrary "counterclockwise" orientation for the sides of each face; different choices of orientation lead to different but equivalent polygonal schemata. If we are lucky, we may find a polygonal schema where $\mathit{fsign}(s) = +1$ for every side s; we call such a polygonal schema ***oriented***. An oriented polygonal schema defines a consistent

oriented polygonal schema
orientable surface map
bipartite

clockwise orientation for every face in the map, and thus intuitively for the entire underlying surface.

Equivalently, when we define a signed rotation system for a given surface map, we must choose an arbitrary "clockwise" orientation for the darts entering each vertex; different choices of orientation lead to different but equivalent rotation systems. If we are lucky, we may find an ***oriented rotation system***, meaning a rotation system where $rsign(d) = +1$ for every dart d. An oriented rotation system defines a consistent counterclockwise orientation for every vertex in the map, and thus intuitively for the entire underlying surface.

A surface map is ***orientable*** if it can be described either by an oriented polygonal schema or by an orientable rotation system. Not surprisingly, these two conditions for orientability are equivalent; the equivalence follows from the following lemma by duality. A reflection system (Φ, a, b, c) is ***bipartite*** if its blades Φ can be partitioned into two sets Φ^- and Φ^-, so that each of the involutions a, b, and c match blades in Φ^- to blades in Φ^+. Equivalently, a reflection system for Σ is bipartite if the vertices and edges of the band decomposition $\Sigma^\square$ form a bipartite graph.

Lemma 6.11. *Σ is the map of an oriented polygonal schema if and only if the reflection system of Σ is bipartite.*

Proof: Fix an oriented polygonal schema $\Pi = (S, next, flip, fsign)$. Let Ξ be the reflection system defined by our earlier rules (6.1), and let $\Phi^+ = \{(s, +1) \mid s \in S\}$ and $\Phi^- = \{(s, -1) \mid s \in S\}$. Because $fsign(s) = +1$ for every side s, we have $a(b) = (flip(s), -\varepsilon)$ for every side s and sign ε. As in any other reflection system, we have $b(s, \varepsilon) = (next^\varepsilon(s), -\varepsilon)$ and $c(s, \varepsilon) = (s, -\varepsilon)$ for every side s and sign ε. We conclude that Ξ is bipartite, because each involution matches positive blades in Φ^+ to negative blades in Φ^-.

Conversely, let $\Xi = (\Phi, a, b, c)$ be a bipartite reflection system. Let Π be the rotation system defined by our earlier rules (6.3), where Φ^+ and Φ^- be the two sides of the bipartition. For any positive blade $\phi^+ \in \Phi^+$, we have $c(\phi^+) \in \Phi^-$, and thus $rsign(\phi^+) = +1$. We conclude that Π is an *oriented* rotation system. □

- Two orientations; primal and dual orientations are opposite.
- Orientable iff every cycle has odd number of twisted edges.
- Oriented spanning tree/spanning cotree

Lemma 6.12. *A surface map Σ is non-orientable if and only if its underlying surface $|\Sigma|$ contains a Möbius band.*

Proof (sketch): First, suppose Σ is non-orientable. Then in any signed rotation system for Σ, there must be a cycle such that the product of the *rsigns* of the edges is negative. Let M be the union of faces in $\Sigma^\square$ corresponding to the vertices and edges of this cycle. One can mechanically verify that M is a Möbius band.

barycentric subdivision
Bary Sigma
flag
One can mechanically verify
common refinement
radial map
Radial Sigma

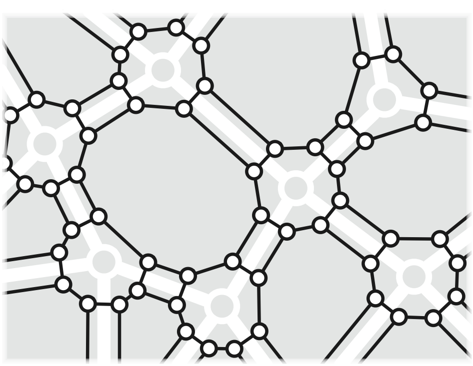
On the other hand, suppose $M \subseteq |\Sigma|$ is a Möbius band. Without loss of generality, no vertex of Σ lies within M. Let $f_1, e_1, f_2, \ldots, e_k$ be the circular sequence of edges and faces that M crosses. Then in any signed polygonal schema for Σ, we have $\prod_i \mathit{fsign}(e_i) = -1$. In particular, we must have $\mathit{fsign}(e) = -1$ for at least one edge e of Σ. Thus, Σ cannot be described by an oriented polygonal schema. $\square$

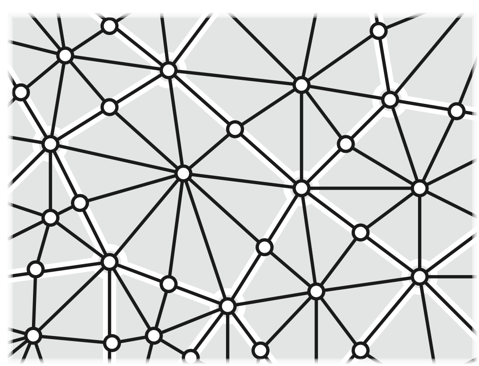
6.5 Other Derived Maps

Barycentric Subdivision

The ***barycentric subdivision*** $\boldsymbol{\Sigma^+}$ of a surface map Σ is the dual of the band decomposition $\Sigma^\square$. The barycentric subdivision can be obtained directly by partitioning the faces of Σ into triangles, called ***flags***, as follows. For each face f of Σ, fix an arbitrary interior point f^+, called the *centroid* of f. Similarly, for each edge e of Σ, fix an arbitrary interior point e^+, called the *midpoint* of e. Each flag is a triangle joining the centroid of some face, the midpoint of some edge incident to that face, and one of the endpoints of that edge. Thus, any face with k sides is partitioned into $2k$ flags in Σ^+; symmetrically any vertex with degree d is surrounded by $2d$ flags in Σ^+.

One can mechanically verify that dual maps have identical barycentric subdivisions: $(\Sigma^*)^+ = \Sigma^+$. Moreover, Σ^+ is a ***common refinement*** of Σ and Σ^*; that is, each face of Σ and each face of Σ^* is the union of vertices, edges, and faces of Σ^+. This observation yields an even easier proof that the maps Σ and Σ^* (and therefore also Σ^+ and $\Sigma^\square = (\Sigma^+)^*$) have the same underlying surface.


The band decomposition $\Sigma^\square$


The barycentric subdivision Σ^+

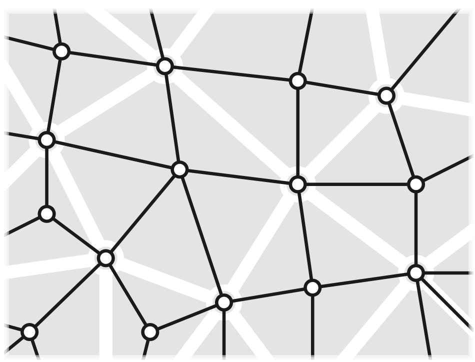
Radial and Medial Maps

Deleting all the midpoint vertices of the barycentric subdivision yields the ***radial map*** of Σ [1, 71], which we denote $\boldsymbol{\Sigma^\diamond}$. Equivalently, we can construct $\Sigma^\diamond$ from Σ by adding a new vertex $f^\diamond$ in the interior of each face f of Σ, adding edges between each

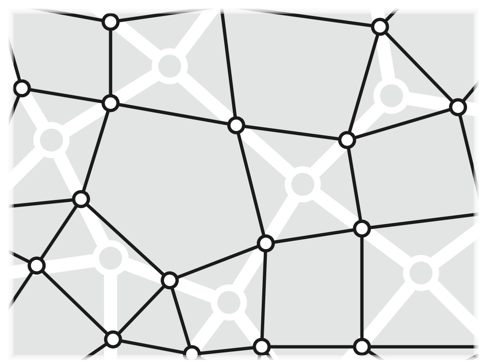
medial map
Medial Sigma

centroid $f^\diamond$ and the original vertices of the corresponding face f, and finally deleting the original edges of Σ. Dual maps share the same radial graph: $\Sigma^\diamond = (\Sigma^*)^\diamond$. By construction, the medial graph is bipartite and every face is a quadrilateral; conversely, any bipartite surface map with quadrilateral faces is the radial map of some dual pair of maps.

The dual of the radial map is the ***medial map*** $\Sigma^\times$ [1,65,83]. The vertices of $\Sigma^\times$ are the "midpoints" of edges of Σ; the edges of $\Sigma^\times$ correspond to pairs of edges in Σ that are consecutive around some face, or equivalently, around some vertex. Alternatively, we can construct the medial map by contracting all a and c edges in the band decomposition $\Sigma^\square$. By construction, every vertex of the radial map has degree 4. Again, dual maps share the same medial graph: $\Sigma^\times = (\Sigma^*)^\times$.



The radial map $\Sigma^\diamond$



The medial map $\Sigma^\times$

My suggested notations $\Sigma^\square$, Σ^+, $\Sigma^\diamond$ and $\Sigma^\times$ are nonstandard, but intended to be mnemonic; the superscript symbol resembles the feature in the derived map corresponding to a horizontal or vertical edge of Σ. The barycentric subdivision is more commonly denoted either Σ' or $Sd\,\Sigma$; the other derived maps have no standard notation.[7]

The following table summarizes the correspondences between features of any map Σ and features of the derived maps we have described so far.

Σ	Σ^*	$\Sigma^\square$	Σ^+	$\Sigma^\times$	$\Sigma^\diamond$
vertex	face	face	vertex	face	vertex
edge	edge	face	vertex	vertex	face
face	vertex	face	vertex	face	vertex
dart	side	edge	edge	corner	corner
corner	corner	edge	edge	edge	edge
side	dart	edge	edge	corner	corner
blade	blade	vertex	face	dart	side

Correspondences between features of related maps

6.6 Data Structures

self-dual incidence list
sorted incidence lists
graph-encoded map
generalized map
cell-tuple
quad-edge

Any data structure for planar graphs can actually be used to represent arbitrary orientable surface maps, with no modifications. In particular, the self-dual incidence list data structure described in Section **??** simultaneously represents a rotation system and a polygonal schema for any orientable surface map.

More effort is required to represent non-orientable surface maps. Some planar data structures require only minor modifications; for example, sorted incidence lists require only one additional bit $rsign(d)$ in each dart record to represent a signed rotation system, or (via duality) a signed polygonal schema. However, there is no analogue of the self-dual incidence list for non-orientable maps. We can of course store any map Σ and its dual Σ^* in two independent (signed) half-edge data structures, with pointers connecting the *four* records for each edge, but because we must distinguish between darts and sides, we cannot merge the two data structures when Σ is (or might become) non-orientable.

An arguably simpler option is to use a blade-based data structure. The *graph-encoded map* representation of Lins [55], Lienhardt's *generalized maps* [52, 53], and Brisson's *cell-tuple* structure [8] are all essentially equivalent to a reflection system. Each of these data structures stores a record for each blade ϕ with pointers to the adjacent blades $a(\phi)$, $b(\phi)$, and $c(\phi)$; each vertex and face is also represented by a pointer to one of its blades. All three structures can be viewed as signed, sorted incidence lists for the band decomposition $\Sigma^\square$, simplified by the fact that every vertex of $\Sigma^\square$ has degree 3 and any two adjacent vertices store their outgoing edges in opposite orders.

Guibas and Stolfi's *quad-edge* data structure [34] uses a slightly different approach. The quad-edge data structure maintains separate records for the blades of the represented surface map Σ and its dual Σ^*; thus, at least at the interface level, the data structure actually maintains *eight* records per edge.

- Finish Quad-edge ≈ incidence list for radial map; see also Kirkman [46]!
- We can convert from any data structure to any other in $O(m)$ time, so we don't have to nail down the phrase "Given a surface map...".

Theorem 6.13. *Given a surface map Σ with m edges, we can determine in $O(m)$ time whether Σ is orientable.*

puncture
Sigma bullet

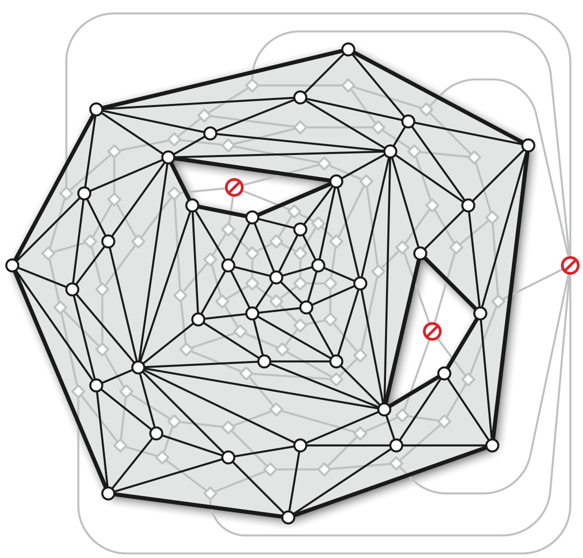
6.7 Surfaces with Boundary

THIS SECTION NEEDS WORK!

- Combinatorial surface minus some faces (or more generally, a dual subcomplex)
- Mark missing faces in data structure
- To avoid boundary cases: delete ***open*** faces $f^\square$ from $\Sigma^{\bullet\square}$, not faces of the original map $\Sigma^\bullet$. In particular, deleting all faces still leaves a 2-manifold with boundary (the ribbon graph).
- Formally, dual of surface with boundary is a punctured surface: delete vertices from surface map (or ***closed*** faces $v^\square$ from $\Sigma^{\bullet\square}$)
- System of arcs $\rightleftharpoons$ System of loops

Duality Again

Any surface map Σ with boundary also has a dual map Σ^*, defined as follows. As usual, the dual map Σ^* has a vertex f^* for each face f of Σ, *including the boundary cycles*, and an edge e^* for each edge e in Σ (including boundary edges) joining the vertices dual to the faces that e separates. For each boundary cycle δ of Σ, we refer to the corresponding vertex δ^* of Σ^* as a ***puncture***; these vertices are marked as missing in any data structure representing Σ^*.



A surface map Σ with boundary (black lines) and its dual map Σ^* (gray lines).
Punctures in the dual map are indicated by $\oslash$.

Equivalently, let $\Sigma^\bullet$ denote the surface map without boundary obtained from Σ by gluing a disk to each boundary cycle (or equivalently, ignoring the flags indicating which faces are missing). The dual map Σ^* is obtained from $(\Sigma^\bullet)^*$ by "deleting" the puncture vertices (or equivalently, setting flags indicating those vertices as missing).

tree-coforest decomposition
forest-cotree decomposition

Formally, the underlying spaces $|\Sigma|$ and $|\Sigma^*|$ are *not* homeomorphic. Recall that $|\Sigma^\bullet|$ is a compact 2-manifold without boundary. The primal space $|\Sigma|$ is a compact 2-manifold *with* boundary, obtained from $|\Sigma^\bullet|$ by deleting a finite number of open disks; whereas, the dual space $|\Sigma^*|$ is a *non-compact* 2-manifold without boundary, obtained from $|\Sigma^\bullet|$ by deleting a finite set of points (or equivalently, *closed* disks). However, as the two underlying spaces are homotopy-equivalent, we can often (but not always!) harmlessly ignore this distinction.

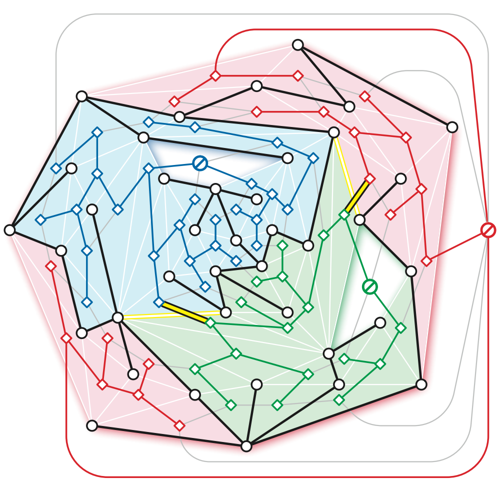
Tree-Coforest and Forest-Cotree Decompositions

Tree-cotree decompositions have two natural generalizations to surface maps with boundary, which are essentially dual to each other. A ***tree-coforest decomposition*** of a map Σ is a partition (T, A, F) of the edges of Σ into three edge-disjoint subgraphs with the following properties:

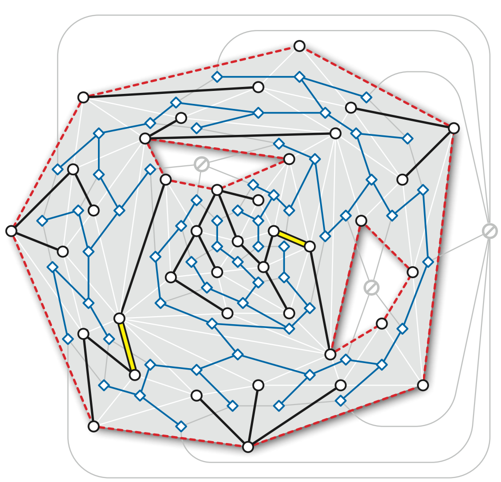
- T is a spanning tree of Σ.
- F^* is a spanning *forest* of the dual map Σ^*, that is, an acyclic subgraph that contains every vertex of Σ^*, including the punctures.
- Each component of F^* contains a single puncture.

Similarly, a ***forest-cotree decomposition*** of Σ is a partition $(\partial\Sigma, F, A, C)$ of the edges of Σ into *four* edge-disjoint subgraphs with the following properties:

- $\partial\Sigma$ is the set of all boundary edges of Σ.
- F is a spanning forest of Σ, that is, an acyclic subgraph that contains every vertex.
- Each component of F contains a single boundary vertex of Σ.
- C^* is a subtree of Σ^* that contains every vertex *except* the punctures.



A tree-coforest decomposition.
Doubled lines indicate edges in A^*;
colors indicate components of F^*.



A forest-cotree decomposition.
Doubled lines indicate edges in A.

Notes

1. (page 1) Formally, to avoid pathological spaces like Alexandrov's long line and the line with two origins [77], manifolds must also be Hausdorff and second-countable, but recall from the introduction that all topological spaces in this book are assumed to be well-behaved! On the other hand, we do *not* require manifolds to be piecewise-linear or differentiable *a priori*.

2. (page 2) Most historical sources observe that described the Möbius strip in July 1858, two months before its first appearance in Möbius's private notes, and first published it five years later [57]. However, Listing merely mentions the one-sided strip in passing, in a footnote, stating only that "such surfaces have very different properties"; whereas, Möbius described the one-sided strip in much more detail, giving both a formal construction from five triangles and an intuitive construction from twisted paper strip, as part of the first formal investigation of non-orientable surfaces.

On the other hand, a construction of the Möbius strip is already *implicit* in another paper of Listing, published more than a decade earlier [56, pp. 857–858]. Listing observed that if two threads are twisted around each other and their ends tied together, the result is either two (possibly linked) cycles or one (possibly knotted) cycle, depending on whether the threads are twisted an integral or half-integral number of times. Tait [78] eloquently described a popular application of Listing's observation:

> The consideration of double-threaded screws, twisted bundles of fibres, &c., leads to the general theory of paradromic winding. From this follow the properties of a large class of knots which form "clear coils." A special example of these, given by Listing for threads, is the well-known juggler's trick of slitting a ring-formed band up the middle, through its whole length, so that instead of separating into two parts, it remains in a continuous ring. For this purpose it is only necessary to give a strip of paper one half-twist before pasting the ends together. If three half-twists be given, the paper still remains a continuous band after slitting, but it cannot be opened into a ring, it is in fact a trefoil knot. This remark of Listing's forms the sole basis of a work which recently had a large sale in Vienna: - showing how, in emulation of the celebrated Slade, to tie an irreducible knot on an endless string!

Henry Slade was an 19th century American spiritualist who claimed, among his many supernatural talents, the ability to introduce knots into a string whose ends had been tied together and covered with wax. When he was arrested for fraud in London, Slade was defended several prominent physicists, who argued that he was passing the string into the fourth dimension [76, 91, 92]. The Vienna street-magic trick described by Tait has been improved and generalized many times by magicians, most commonly under the rubric "The Afghan Bands". Perhaps the most famous version of the Afghan Bands was designed by James C. Wobensmith and performed by Harry Blackstone, Sr. using prepared strips of muslin [30].

Where was I? Oh, right. On the gripping hand, Möbius and Listing both studied mathematics under Gauss, and some authors have suggested that Gauss may have described the one-sided band to both of his students before 1958 [41, 67]. Indeed, in his first article on topology, Listing credits “den größten Geometer der Gegenwart” (obviously his advisor!) for drawing his attention to the importance of the field [56, p. 813]. Gauss himself never published in topology.

3. (page 2) The word “map” unfortunately has at least two distinct but completely standard meanings in topology. The most common meaning is a function that preserves some interesting structure (although some subfields still use the older term “mapping”). In particular, topologists commonly use “map” to describe any continuous function, or (as a verb) the action of that function; this usage leads to derived terms like “covering map”, “mapping torus”, and “mapping class group”. The use of “map” to denote a cellular surface decomposition dates back to the earliest statements of the four-color conjecture, motivated by Francis Guthrie’s attempt to color an actual map of the counties of England [79]. This homonymy can be confusing, especially when discussing continuous functions between cellular decompositions, but there is no choice but to tread carefully.

4. (page 5) Perhaps the first and most famous *fictional* signed polygonal schema is Lewis Carroll’s description of the Purse of Fortunato [9], quoted at the beginning of this chapter. The passage continues as follows:

> “You shall first,” said Mein Herr, possessing himself of two of the handkerchiefs, spreading one upon the other, and holding them up by two corners, “you shall first join together these upper corners, the right to the right, the left to the left; and the opening between them shall be the *mouth* of the Purse.”
>
> A very few stitches sufficed to carry out *this* direction. “Now, if I sew the other three edges together”, she suggested, “the bag is complete?”
>
> “Not so, Miladi: the *lower* edges shall first be joined— ah, not so!” (as she was beginning to sew them together). “Turn one of them over, and join the right lower corner of the one to the *left* lower corner of the other, and sew the lower edges together in what you would call *the wrong way*.”
>
> “I see!” said Lady Muriel, as she deftly executed the order. “And a very twisted, uncomfortable, uncanny-looking bag it makes! But the *moral* is a lovely one. Unlimited wealth can only be attained by doing things *in the wrong way!* And how are we to join up these mysterious—no, I mean this mysterious opening?” (twisting the thing round and round with a puzzled air). “Yes, it is one opening. I thought it was *two*, at first.”
>
> “You have seen the puzzle of the Paper Ring?” Mein Herr said, addressing the Earl. “Where you take a slip of paper, and join its ends together, first twisting one, so as to join the *upper* corner of *one* end to the *lower* corner of the *other?*”
>
> “I saw one made, only yesterday,” the Earl replied. “Muriel, my child, were you not making one, to amuse the children you had to tea?”
>
> “Yes, I know that Puzzle,” said Lady Muriel. “The Ring has only *one* surface, and only *one* edge, It’s very mysterious!”
>
> “The bag is just like that, isn’t it?” I suggested. “Is not the outer surface of one side of it continuous with the inner surface of the other side?”

projective plane

"So it is!" she exclaimed. "Only it *isn't* a bag, just yet. How shall we fill up this opening, Mein Herr?"

"Thus!" said the old man impressively, taking the bag from her, and rising to his feet in the excitement of the explanation. "The edge of the opening consists of *four* handkerchief edges, and you can trace it continuously, round and round the opening: down the right edge of *one* handkerchief, up the left edge of the *other*, and then down the left edge of the *one*, and up the right edge of the *other!*"

"So you can!" Lady Muriel murmured thoughtfully, leaning her head on her hand, and earnestly watching the old man. "And that *proves* it to be only *one* opening!"

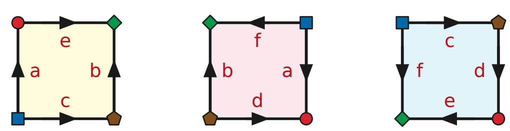
She looked so strangely like a child, puzzling over a difficult lesson, and Mein Herr had become, for the moment, so strangely like the old Professor, that I felt utterly bewildered: the "eerie" feeling was on me in its full force, and I felt almost *impelled to say* "Do *you* understand it, Sylvie?" However I checked myself by a great effort, and let the dream (if indeed it was a dream) go on to its end.

"Now, this *third* handkerchief", Mein Herr proceeded "has *also* four edges, which you can trace continuously round and round: all you need do is to join its four edges to the four edges of the opening. The Purse is then complete, and its outer surface—"

"I see!" Lady Muriel eagerly interrupted. "Its *outer* surface will be continuous with its *inner* surface! But it will take time. I'll sew it up after tea." She laid aside the bag, and resumed her cup of tea. "But why do you call it Fortunatus's Purse, Mein Herr?"

The dear old man beamed upon her, with a jolly smile, looking more exactly like the Professor than ever. "Don't you see, my child—I should say Miladi? Whatever is *inside* that Purse, is *outside* it; and whatever is *outside* it, is *inside* it. So you have all the wealth of the world in that leetle Purse!"

His pupil clapped her hands, in unrestrained delight. "I'll certainly sew the third handkerchief in—*some* time," she said: "but I wo'n't take up your time by trying it now. Tell us some more wonderful things, please!"



The Purse of Fortunatus

The Purse of Fortunatus is a map of the projective plane, the simplest non-orientable surface without boundary. The hapless Victorian lady who attempted to follow Mein Herr's instructions would soon discover that the third handkerchief cannot actually be sewn to the other two, except perhaps with the help of the celebrated Slade.

5. (page 7) A map is regular if it is connected and has a blade-transitive symmetry group; that is, for any two blade of Σ (defined in the next section), there is a unique homeomorphism from Σ to itself that sends one blade to the other. The regular planar maps are precisely the Platonic solids, the dihedra (two faces bounded by a common subdivision surfaces

cycle), the hosohedra (two vertices connected by parallel edges), and the trivial map (which has no blades).

The complete classification of regular maps is an open problem; it is not even known precisely which surfaces support regular maps. The sphere, the projective plane, and the torus are the only surfaces that support an infinite number of distinct regular maps. Every orientable surface supports at least one regular map (with one vertex and once face). Infinitely many non-orientable surfaces support regular maps [11], but also infinitely many (including the Klein bottle) do not [18]. Conder has classified all maps on surface with Euler characteristic at least −200 [12]. Širáň [74, 75] provides accessible surveys of the state of the art.

6. (page 10) To make the terminology even more confusing, many authors, including Jones and Singerman [44] and Tutte [82], also use the phrase "combinatorial map" to denote either a polygonal schema, a rotation system, or a permutation group (generated either by an involution and another permutation or by three involutions).

7. (page 20) Mohar and Thomassen [61] write R_G and M_G to respectively denote the radial and medial graphs of a surface-embedded graph G.

Conway proposed a family of symmetry-preserving operations on polyhedra [13], which can actually be applied to any surface map. In Conway's notation, the dual of Σ is denoted $d\Sigma$, the band decomposition is $b\Sigma$ ("bevel"), the barycentric subdivision is $m\Sigma$ ("meta"), the medial map is $a\Sigma$ ("ambo"), and the radial map is $j\Sigma$ ("join").

Pisanski and Randić [66] proposed different notation for a similar set of standard map operations, motivated by the study of fullerenes; Mallos [59] later extended their notation by incorporating several of Conway's operations, as a vocabulary for mathematical sculpture. In Mallos' notation, the dual of Σ is denoted $Du(\Sigma)$, the band decomposition is $Be(\Sigma)$ ("bevel"), the barycentric subdivision is $Mt(\Sigma)$ ("meta"), the medial map is $Me(\Sigma)$, and the radial map is $Ra(\Sigma)$.

Similar map operations are also applied recursively to define subdivision surfaces in computer graphics [10, 20, 21, 35, 49, 58, 62, 64, 93].

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