handle detach a handle separating cycle non-separating cycle two-sided cycle

## CHAPTER

## Surface Classification

Poincaré's ignorance of the mathematical literature, when he started his researches, is almost unbelievable.

- Jean Dieudonné (1975)
describing Poincaré's early work on Riemannian surfaces

The main interest in Jordan's attempt [to prove the classification of compact orientable surfaces in $\mathbb{R}^{3}$ ] is in showing how the work of an outstanding mathematician can appear nonsensical a century later.

- Morris Hirsch, Differential Topology (1976)

Short paragraph describing the classification theorem

### 7.1 Handles

A handle in a surface $\mathcal{S}$ is a non-separating annulus, that is, an annulus $A$ whose complement $\mathcal{S} \backslash A$ is connected. The Jordan curve theorem implies that the sphere has no handles, but this theorem does not extend to other surfaces. To detach the handle, we delete it from $\mathcal{S}$ and glue disks to the two resulting boundary circles, as shown in the figure below.

Now let $\Sigma$ be a map on some surface $\mathcal{S}$. Handles may appear in any map $\Sigma$ as the union of faces, but to avoid special cases and boundary conditions, it is more convenient to consider the band decomposition $\Sigma^{\square}$. Recall that $v^{\square}$, $e^{\square}$, and $f^{\square}$ denote the faces of $\Sigma^{\square}$ corresponding to vertex $v$, edge $e$, and face $f$, respectively.

A cycle of vertices and edges in $\Sigma$ is separating if its complement is disconnected, and non-separating otherwise. A cycle is two-sided if it has a neighborhood homeo-
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Left to right: Detaching a handle by deleting an isthmus.
Right to left: Attaching a handle by inserting an edge between corners of different faces.
morphic to an annulus; equivalently, if $\Sigma$ is represented by a signed rotation system, a cycle is two-sided if it has an even number of edges $e$ with $r \operatorname{sign}(e)=-1$.

Any non-separating two-sided cycle in $\Sigma$ corresponds to a sequence of faces in $\Sigma^{\square}$ whose union is a handle in $\mathcal{S}$. This handle can be detached by contracting the edges of the cycle in arbitrary order. Each contraction except the last changes the map but leaves the underlying surface $\mathcal{S}$ unchanged. The last edge $e$ to be contracted is a non-separating loop. If we ignore our earlier proscription against contracting loops, then contracting $e$ (as described by Equation ??) actually detaches the handle $e^{\square} \cup v^{\square}$, as shown below; in particular, the contraction splits $v$ into two vertices. (If $e$ were a separating loop, this contraction would actually disconnect the surface.)


Symmetrically, any non-separating two-sided cycle in the dual map $\Sigma^{*}$ can be viewed as a cyclic sequence of edges and faces in $\Sigma$; the union of the corresponding faces in $\Sigma^{\square}=\left(\Sigma^{*}\right)^{\square}$ is also a handle in $\mathcal{S}$. This handle can be detached by deleting the edges of the cocycle in arbitrary order, using the algorithm described by Equation ??. The last edge $e$ to be deleted is a non-bridge isthmus incident to some face $f$; deleting $e$ (as described by Equation ??) actually detaches the handle $e^{\square} \cup f^{\square}$. (Deleting a bridge would actually disconnect the surface.)

The inverse operation-deleting two open disks and gluing an annulus onto the resulting boundary circles-is called attaching a handle. There are two different ways to attach a handle to an orientable surface $\mathcal{S}$, depending on how the boundary circles on the surface are oriented. Let's assume that the boundary circles of the annulus are oriented so that the actual surface is always to the left. If the boundary circles on $\mathcal{S}$
have the same orientation, as shown on the previous page, the resulting surface is still orientable. If they have opposite orientations, as shown below, the resulting surface is non-orientable; in this case, we say that the handle is disorienting. For non-orientable surfaces, there is no consistent way to define the orientation of the boundary circle, so "both" ways of attaching a handle are actually equivalent.


Attaching or detaching a disorienting handle.
We can attach a handle to an surface map $\Sigma$ either by inserting an edge between corners of two different faces of $\Sigma$ or by "expanding" an edge between corners of two different vertices of $\Sigma$. In the first case, the new edge becomes a non-bridge isthmus; in the second case, the new edge becomes a non-separating loop. Both edge operations can be performed in two different ways, depending on which pairs of blades are connected. Specifically, when we insert an edge $e$, we must choose $\operatorname{rsign}(e)$; when we expand an edge $e$, we must choose $f \operatorname{sign}(e)$. If the surface map $\Sigma$ is orientable, only one of these two choices preserves orientability. If $\Sigma$ is non-orientable, the two choices still yield two different maps on the same non-orientable surface.

### 7.2 Twists

Recall that a 2-manifold $S$ is non-orientable if and only if it has a subspace homeomorphic to the Möbius band; we call such a subspace a twist of $\mathcal{S}$. Because a twist has only one boundary cycle, every twist is non-separating. To detach the twist, we delete it from $\mathcal{S}$ and then glue a disk onto the resulting boundary circle, as shown in the figure below.


Let $\Sigma$ be a map on a non-orientable surface $\mathcal{S}$. A cycle of vertices and edges in $\Sigma$ is
one-sided cycle self-dual twist
one-sided if it has a neighborhood homeomorphic to a Möbius band, or equivalently, if it is not two-sided. Any one-sided cycle in $\Sigma$ corresponds to a sequence of faces in the band decomposition $\Sigma^{\square}$ whose union is a twist in $\mathcal{S}$. This twist can be detached by contracting the edges of the cycle in arbitrary order, again following Equation ??. The last edge $e$ to be contracted is a one-sided loop incident to some vertex $v$; contracting this loop actually detaches the twist $e^{\square} \cup v^{\square}$, as shown below. Note that $v$ is still a vertex in the resulting map, but with the remaining incident darts in a different order.


Symmetrically, deleting the edges in $\Sigma$ dual to a one-sided cycle in $\Sigma^{*}$ detaches a twist from the underlying surface. The last edge to be deleted is a twisted isthmus incident to some face $f .{ }^{1}$ After this edge is deleted, $f$ is still a face in the resulting map, but with its incident sides in different order.

It is possible for a single edge to be both a one-sided loop and a twisted isthmus; we call such an edge a self-dual twist. The maps obtained by deleting and contracting a self-dual twist are actually identical.


Contracting or deleting a self-dual twist.

The inverse operation-deleting a disk and gluing an Möbius band onto the resulting boundary circle-is called attaching atwist. The resulting surface is always nonorientable. We can attach a twist to an surface map $\Sigma$ either by inserting an edge $e$ with $f \operatorname{sign}(e)=-1$ between two corners of the same face, or by expanding an edge $e$ with $\operatorname{rsign}(e)=-1$ between two corners of the same vertex.

### 7.3 The Classification Theorem

For any non-negative integers $g$ and $h$, let $\boldsymbol{\mathcal { S }} \mathbf{g}, \mathbf{0})$ denote the orientable surface obtained from the sphere by attaching $g$ handles. For example, $\mathcal{S}(0,0)$ is the sphere, and $\mathcal{S}(1,0)$ is the torus. We leave the proof that these surfaces are well-defined up to homeomorphism as an exercise for the reader; it does not matter where or in what order the handles are attached, as long as none of the handles is disorienting. ${ }^{2}$

Lemma 7.1. Every orientable surface map lies on the surface $\mathcal{S}(g, 0)$ for some integer $g \geq 0$.

Proof: Fix an orientable surface map $\Sigma$. For any tree-cotree decomposition ( $T, L, C$ ) of $\Sigma$, the map $\Sigma / T \backslash C$ is a system of loops on the same underlying surface as $\Sigma$. Thus, without loss of generality, we can assume that $\Sigma$ itself is a system of loops. Because $\Sigma$ is orientable, every loop in $\Sigma$ is two-sided.

If $\Sigma$ has no edges, it must be the trivial map on the sphere $\mathcal{S}(0,0)$. Otherwise, let $\ell$ be an arbitrary edge of $\Sigma$. Because $\Sigma$ has only one vertex and one face, $\ell$ is a non-separating loop. Thus, we can detach a handle from $|\Sigma|$ by contracting $\ell$. The induction hypothesis implies that $|\Sigma / \ell|=\mathcal{S}\left(g^{\prime}, 0\right)$ for some integer $g^{\prime} \geq 0$. We conclude that $|\Sigma|=S\left(g^{\prime}+1,0\right)$.

Similarly, for any integers $g \geq 0$ and $h>0$, let $\mathcal{S}(g, h)$ denote the surface obtained from the sphere by attaching $g$ handle and $h$ twists. It does not matter where or in what order the handles and twists are attached, or whether any of the handles is disorienting. Again, we leave the proof that $\mathcal{S}(g, h)$ is well-defined up to homeomorphism as an exercise for the reader. For example, $\mathcal{S}(0,1)$ is the projective plane, and $\mathcal{S}(0,2)$ is the Klein bottle.

Lemma 7.2. Every non-orientable surface map lies on the surface $\mathcal{S}(g, h)$ for some integers $g \geq 0$ and $h \geq 1$.

Proof: Fix a non-orientable surface map $\Sigma$; as in the proof of Lemma 7.1, we can assume without loss of generality that $\Sigma$ is a system of loops. Let $\ell$ be an arbitrary one-sided loop in $\Sigma$; such a loop must exist because $\Sigma$ is non-orientable. We can detach a twist from $|\Sigma|$ by contracting $\ell$. If the map $\Sigma / \ell$ is orientable, then $|\Sigma / \ell|=\mathcal{S}(g, 0)$ for some integer $g \geq 0$ by Theorem ??; it follows that $|\Sigma|=\mathcal{S}(g, 1)$. Otherwise, the
attaching a twist Surface g 0 exercise for the reader exercise for the reader projective plane Klein bottle

Dyck's surface genus
induction hypothesis implies that $|\Sigma / \ell|=\mathcal{S}\left(g, h^{\prime}\right)$ for some integers $g \geq 0$ and $h \geq 1$; it follows that $|\Sigma|=\mathcal{S}\left(g, h^{\prime}+1\right)$. In both cases, the proof is complete.

The proofs of Lemmas 7.1 and 7.2 imply the following simple algorithm to classify the underlying surface of a given surface map:

```
\(\underline{\operatorname{ClASSIFY}(\Sigma):}\)
    \((T, L, C) \leftarrow\) any tree-cotree decomposition of \(\Sigma\)
    \(\Sigma \leftarrow \Sigma / T \backslash C\)
    \((g, h) \leftarrow(0,0)\)
    while \(\Sigma\) non-orientable
        \(\ell \leftarrow\) any one-sided loop in \(\Sigma\)
        \(\Sigma \leftarrow \Sigma / \ell\)
        \(h \leftarrow h+1\)
    while \(\Sigma\) is non-trivial
        \(\ell \leftarrow\) any loop in \(\Sigma\)
        \(\Sigma \leftarrow \Sigma / \ell\)
        \(e \leftarrow\) any non-loop edge in \(\Sigma\)
        \(\Sigma \leftarrow \Sigma / e\)
        \(g \leftarrow g+1\)
    return \((g, h)\)
```

However, this algorithm may output different classifications for the same surface map, depending on which edge is contracted in each iteration. Consider the following example, called Dyck's surface [11]. Let $\Sigma$ be a system of three one-sided loops $x, y, z$ incident to the unique vertex in the order $x, y, z, x, y, z$. Contracting $x$ gives us an orientable system of loops on the torus $\mathcal{S}(1,0)$, implying that $|\Sigma|=S(1,1)$. On the other hand, contracting edge $y$ yields a non-orientable system of loops on the Klein bottle $\mathcal{S}(0,2)$, implying that $|\Sigma|=\mathcal{S}(0,3)$. We conclude that $\mathcal{S}(1,1)=\mathcal{S}(0,3)$.

A straightforward inductive argument now implies the following more general equivalence, which in turn implies a simpler classification of non-orientable surfaces.

Lemma 7.3 (Dyck [11]). $\mathcal{S}(g, h)=\mathcal{S}(0, h+2 g)$ for all positive integers $g$ and $h$.

Theorem 7.4 (Classification of Surface Maps). Every connected surface map lies on either $\mathcal{S}(g, 0)$ or $\mathcal{S}(0, g)$, for some integer $g \geq 0$.

The integer $g$ is called the genus of both the surface map and the underlying surface. The genus of a surface $\mathcal{S}$ can be equivalently defined as the maximum number of disjoint simple cycles $\gamma_{1}, \gamma_{2}, \ldots, \gamma_{g}$ in $\mathcal{S}$ such that the complement $\mathcal{S} \backslash\left(\gamma_{1} \cup \gamma_{2} \cup \cdots \cup \gamma_{g}\right)$ is still connected. If $\mathcal{S}$ is orientable, these cycles are necessarily two-sided; if $\mathcal{S}$ is non-orientable, the proof of the classification theorem implies that every cycle $\gamma_{i}$ is one-sided.


A band decomposition of Dyck's surface $\mathcal{S}(1,1)=\mathcal{S}(0,3)$.

## 7.4 "Oilers' Formula"

The Euler characteristic $\boldsymbol{\chi}(\boldsymbol{\Sigma})$ of a surface map $\Sigma=(V, E, F)$ is the number of vertices minus the number of edges plus the number of faces: $\chi(\Sigma)=|V|-|E|+|F|$. The following generalization of Euler's formula for planar graphs implies that the Euler characteristic is actually an invariant of the underlying surface $|\Sigma|$.

Theorem 7.5 (Euler's formula for surface maps). Every map on the surface $\mathcal{S}(g, h)$ has Euler characteristic $2-2 g-h$.

Proof: Let $\Sigma$ be any map on on the surface $\mathcal{S}(g, h)$. Contracting a non-loop edge in $\Sigma$ decreases both the number of vertices and the number of edges by 1 , leaving the Euler characteristic unchanged. Similarly, deleting a non-isthmus edge in $\Sigma$ decreases both the number of edges and the number of faces by 1 , leaving the Euler characteristic unchanged. Thus, without loss of generality, we can assume that $\Sigma$ is a system of loops.

The trivial map clearly has Euler characteristic 2 . Contracting a one-sided loop $\ell_{1}$ yields a new system of loops with one less edge; thus, $\chi\left(\Sigma / \ell_{1}\right)=\chi(\Sigma)-1$. Contracting a two-sided loop $\ell_{2}$ reduces yields a new map with two vertices, one less edge than $\Sigma$, and one face; thus, $\chi\left(\Sigma / \ell_{2}\right)=\chi(\Sigma)-2$. The theorem now follows immediately by induction.

Corollary 7.6. Two connected surface maps lie on the same underlying 2-manifold if and only if (1) they are either both orientable or both non-orientable and (2) their Euler characteristics are equal.

Corollary 7.7. For any tree-cotree decomposition ( $T, L, C$ ) of any map on the surface $\mathcal{S}(g, h)$, we have $|L|=2 g+h$.

Corollary 7.8. Given a surface map $\Sigma$ with $m$ edges, we can determine the homeomorphism class of $|\Sigma|$ in $O(m)$ time.

### 7.5 All Compact 2-Manifolds Support Maps

Most early results in surface topology either implicitly assumed that every compact 2-manifold is the underlying surface of some map or explicitly defined surfaces to be the spaces described by polygonal schemata. The existence of a map for any compact surface was first proved independently by Kerékjártó [16] and Radó [24]; in fact, Kerékjártó proved a more general result about non-compact surfaces. The following proof, which loosely follows Thomassen [28], relies on two technical lemmas that are somewhat easier to prove in isolation.

Lemma 7.9. Let $A$ and $B$ be disjoint closed subsets of a compact space $X$. Any path in $X$ has only a finite number of subpaths that start in $A$, end in $B$, and otherwise lie in $X \backslash(A \cup B)$.

Proof: Fix a path $\pi:[0,1] \rightarrow X$. Call a subpath of $\pi$ evil if the subpath starts in $A$, ends in $B$, and otherwise lies in $X \backslash(A \cup B)$. Suppose to the contrary that $\pi$ has an infinite number of evil subpaths. Then there is an infinite increasing sequence of real values $0<s_{1}<t_{1}<s_{2}<t_{2}<\cdots<1$ such that $\pi\left(s_{i}\right) \in A$ and $\pi\left(t_{i}\right) \in B$ for all $i$. Compactness of $X$ implies that there is a subsequence $0<u_{1}<v_{1}<u_{2}<v_{2}<\cdots<1$ of these values with $\pi\left(u_{i}\right) \in A$ and $\pi\left(v_{i}\right) \in B$ for all $i$, such that the subsequence $\pi\left(u_{1}\right), \pi\left(u_{2}\right), \ldots$ converges to a point in $A$ and the subsequence $\pi\left(v_{1}\right), \pi\left(v_{2}\right), \ldots$ converges to a point in $B$. But this is impossible, because both point sequences converge to the same point $\pi(w)$, where $w=\sup _{j} u_{j}=\sup _{j} v_{j}$.

Lemma 7.10. Any disjoint simple closed curves $\alpha$ and $\beta$ in the plane can be separated by the boundary of a simple polygon.

Proof: Without loss of generality, assume that $\beta$ lies in the unbounded component of $\mathbb{R}^{2} \backslash \alpha$. Fix a positive real number $\varepsilon$ that is less than half the distance between $\alpha$ and $\beta$. Let $\alpha_{\varepsilon}$ be the set of points at distance $\varepsilon$ from $\alpha$, and let $\tilde{\alpha}_{\varepsilon}$ denote the boundary of the unbounded component of $\mathbb{R}^{2} \backslash \alpha_{\varepsilon}$. Compactness implies that the closed curve $\tilde{\alpha}_{\varepsilon}$ can be covered by a finite number of disks of radius $\varepsilon / 10$, each centered on a point of $\tilde{\alpha}_{\varepsilon}$. Sort the centers of those disks in cyclic order around $\tilde{\alpha}_{\varepsilon}$ and connect them by line segments to obtain a closed polygonal curve; the triangle inequality implies that $P$ does not intersect $\alpha$ or $\beta$. This polygon may not be simple, but it has a finite number of self-intersection points. Removing all loops that do not enclose $\alpha$ yields a simple polygon that separates $\alpha$ and $\beta$.

Theorem 7.11 (Kerékjártó [16] and Radó [24]). Every compact, connected 2-manifold is the underlying space of at least one surface map.

Proof: Fix a compact, connected 2-manifold $\mathcal{S}$. For any point $p \in \mathcal{S}$, let $U^{\circ}(p)$ be an open subset of $\mathcal{S}$ that contains $p$ and is homeomorphic to an open disk. Let $Q(p)$ be a closed disk in the interior of $U^{\circ}(p)$, such that $p$ lies in the interior of $Q(p)$. The compactness of $\mathcal{S}$ implies that there is a finite set of points $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ such that the interiors of the disks $Q\left(p_{i}\right)$ cover $\mathcal{S}$. To simplify notation, let $Q_{i}=Q\left(p_{i}\right)$ and $U_{i}^{\circ}=U^{\circ}\left(p_{i}\right)$ for each index $i$.

We inductively construct a finite sequence of closed disks $R_{1}, R_{2}, \ldots, R_{n}$ satisfying the following conditions:

- For each index $i$, the closed disk $Q_{i}$ lies in the interior of $R_{i}$. (Thus, $\bigcup_{i} R_{i}=S$.)
- For each index $i$, the closed disk $R_{i}$ lies in the open disk $U_{i}^{\circ}$.
- For all $i$ and $j$, the intersection $\partial R_{i} \cap \partial R_{j}$ has a finite number of components.

Fix an index $m$ and suppose we have already constructed disks $R_{1}, \ldots, R_{m-1}$. For any index $i<m$, we call each component of $\partial R_{i} \cap U_{m}^{\circ}$ a bad path. A bad path is truly evil if it intersects the disk $Q_{m}$, and merely annoying otherwise. Lemma 7.9 (with $A=Q_{m}$ and $B=S \backslash U_{m}^{\circ}$ ) directly implies that only a finite number of bad paths are truly evil, although there may be infinitely many merely annoying paths.

Let $S_{m}$ be a closed disk in $U_{m}^{\circ}$ that avoids every merely annoying path and contains $Q_{m}$ in its interior. 《|Also, the boundary of $S_{m}$ intersects each truly evil path a finite number of times. Why does such a disk exist?)〉 By the inductive hypothesis, each pair of evil paths intersects only a finite number of times. Thus, the union of the boundary of $S_{m}$ and the truly evil paths is a topological plane graph $\Gamma_{m}$ embedded in $U_{m}^{\circ}$. Theorem ?? implies that there is an isomorphic piecewise-linear plane graph $\Gamma_{m}^{\prime}$ in $U_{m}^{\circ}$ whose outer face is the complement of $S$. The Jordan-Schönflies theorem implies that the homeomorphism from $\Gamma_{m}$ to $\Gamma_{m}^{\prime}$ can be extended to a homeomorphism $h_{m}: S_{m} \rightarrow S_{m}$ such that $h_{m}\left(p_{m}\right)=p_{m}$. Lemma 7.10 implies that $U_{m}^{\circ}$ contains a simple polygon $R_{m}^{\prime}$ whose boundary separates the closed curves $h_{m}\left(\partial Q_{m}\right)$ and $\partial S_{m}$. Finally, let $R_{m}=h_{m}^{-1}\left(R_{m}^{\prime}\right)$. It is easy to check that $\partial R_{m} \cap \partial R_{i}$ has a finite number of components, for any index $i<m$. This completes the construction of the disks $R_{1}, R_{2}, \ldots, R_{n}$.

## Figure?

Now the union of the boundary curves $\partial R_{1}, \partial R_{2}, \ldots, \partial R_{n}$ is topological graph $G$ embedded in $\mathcal{S}$. For all indices $i$ and $j$, each component of $\partial R_{i} \cap \partial R_{j}$ is a common subpath (which may be a single point). The vertices of $G$ are the endpoints of all such common subpaths; the edges of $G$ are subpaths of the curves $\partial R_{i}$ between successive vertices. By construction, $G$ has a finite number of vertices and edges. Every face of $G$ lies inside some neighborhood $U_{i}^{\circ}$ and thus is homeomorphic to a polygon with holes. If necessary, we can add edges to $G$ to ensure that every face is a disk.

Corollary 7.12 (Surface Classification). Every compact, connected surface is homoeomorphic to either $\mathcal{S}(g, 0)$ or $\mathcal{S}(0, g)$, for some integer $g \geq 0$.
bad path
truly evil
merely annoying


Corollary 7.13. Two compact connected 2-manifolds are homeomorphic if and only if (1) they are either both orientable or both non-orientable and (2) their Euler characteristics are equal.

### 7.6 Surfaces with Boundary

Every compact connected surface with boundary is homeomorphic to a compact connected surface without boundary minus a finite set of disjoint open disks.

### 7.7 History

The classification of orientable surfaces began with l'Huillier's investigations of "exceptions" to Euler's formula $V-E+F=2[18,19]$. Specifically, l'Huillier (whose name means "the oiler") proved Theorem 7.5 for the special case of polyhedra with disjoint prismatic tunnels, essentially by inclusion-excusion; in hindsight, l'Huillier's formulation and techniques are remarkably ad hoc. (L'Huillier's original paper [18] considered only polyhedra with one tunnel; the easy generalization to multiple tunnels may be due to Gergonne [19].)

The more general notion of genus originates with Abel's seminal study of complex algebraic curves [1,2], although with a less geometrically intuitive definition. Riemann [25] correctly observed that orientable surfaces can be classified by their "connectivity", but he did not give a complete proof. The classification of orientable polyhedral surfaces by their Euler characteristics follows from Listing's systematic study of spatial complexes [20], later informally summarized by Cayley [7], but neither Listing nor Cayley made this connection. The actual term "genus" ("das Geschlecht") was coined by Clebsch [8].

The first self-contained classification of arbirtary orientable surfaces was given by Möbius [21] using an early version of Morse theory. Specifically, Möbius defined the " $n$th basic form" ("Grundform der $n$ ten Klasse") as a sphere with $n$ disks removed. He then argued that any surface in $\mathbb{R}^{3}$ can be decomposed into a finite collection of basic forms by cutting along generic level sets of a function $h$ from the surface to the reals. Using a sequence of basic combinatorial moves, Möbius showed that the surface can be decomposed into exactly two basic forms, which meet along their boundary cycles; the genus of the surface is one less than the number of boundary cycles of these forms. Finally, Möbius argues that two surfaces are homeomorphic if and only if they have the same genus.

By modern standards, Möbius' proof is incomplete for several reasons. First, the concept of homeomorphism was unknown at the time, and indeed was not completely formalized until well into the 20th century [23]. Instead, Möbius defines an "elementary relationship" ("elementar Verwandschaft") between two surfaces as an adjacencypreserving correspondence between "infinitesimal elements" on the two surfaces. Sec-


Splitting a surface into basic forms along level sets [21].
ond, Möbius' argument requires a function from the surface to the reals with only a finite number of critical values. For the algebraic surfaces that Möbius was considering, finding such a function is straightforward. The existence of such a function for arbitrary 2 -manifolds is equivalent to the Kerékjártó-Radó theorem; a suitable function can be derived from any map on the surface, and vice versa. Finally, the entire argument rest implicitly on the Jordan curve theorem. Modern formulations of Möbius' proof appear in several differential topology textbooks; see, for example, Hirsch [13].

- Other arguments for classifying surfaces were given by Jordan [15] (surfaces with boundary and massive hand-waving), Clifford [9] ("canonical form and dissection"), and Klein [17] ("normal form").
- Euler's formula with genus: Becker [4,5] and Hoppe [14]
- Non-orientable surfaces: first considered by Möbius [22], developed further by Klein [17], classified by Dyck [11]
- More classification: Dehn and Heegaard [10], Alexander [3], Brahana [6] (who credits the normalization algorithm to Veblen).

The Veblen-Brahana proof appears in most topology textbooks, thanks to its appearance in an early textbook of Siefert and Threlfall [26,27]. Several other proofs are known; see especially Thomassen's completely self-contained proof (which includes a proof of the Jordan-Schönflies theorem) [28] and Conway's 'zero irrelevancy' proof [12].

## Notes

1. (page 4) Twisted Isthmus is the name of my next band.
2. (page 5)

Handles don't mean what you think they mean. A tubular neighborhood of the graph of the cube has genus 5, but there is no feature that occurs five times. Figure!

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