There is nothing "below the surface," my faithful friend—absolutely nothing.
— Letter 52, *The Mahatma Letters to A. P. Sinnett* (1882)

5 Surfaces

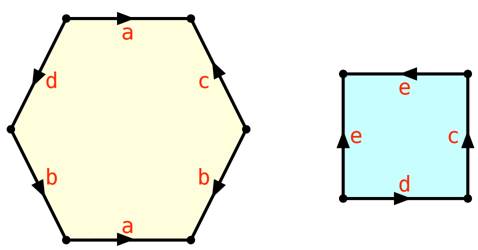
For the next several lectures, we will move from the Euclidean plane to a wider class of topological spaces that locally resemble the plane. A topological space Σ is called a ***2-manifold*** if for every point $x \in \Sigma$, there is an open subset $U \subseteq \Sigma$ such that $x \in U$ and U is homeomorphic to $\mathbb{R}^2$. More succinctly, a 2-manifold is a space that is ***locally homeomorphic*** to the plane. 2-manifolds are also called ***surfaces***.

In this lecture, I'll describe a classical combinatorial description of 2-manifolds that is useful both for abstract mathematical arguments and as the basis of concrete data structure. The same combinatorial structure can be described in two equivalent ways:

- ***Polygonal schema:*** A collection of polygons with edges glued together in pairs.
- ***Cellular graph embedding:*** An embedding of a graph G on a surface Σ, such that every face of the embedding is a topological disk;

5.1 Polygonal Schemata

A ***polygonal schema*** Π is a finite collection of polygons with oriented sides identified in pairs. More explicitly, let $f_1, f_2, \ldots, f_n$ denote a finite set of polygons in the plane, called ***faces***, whose total number of sides is even. (I will always refer to the vertices of these polygons as ***corners***, and their edges as ***segments***.) Formally, an ***orientation*** of an edge e is a linear map from the unit interval $[0,1]$ to e; however, since there are only two such maps, we can think of an orientation of a side e as a permutation of its endpoints, or a labeling of its ***endpoints*** as $head(e)$ and $tail(e)$. A polygonal schema assigns each side an orientation and a ***label*** from some finite set, such that each label is assigned exactly twice. (The standard choice of labels in illustrations are lower-case letters.)



A polygonal schema Π with signature $(\mathtt{a}\bar{\mathtt{b}}\mathtt{c}\bar{\mathtt{a}}\mathtt{db})(\mathtt{cde}\bar{\mathtt{e}})$; arrows indicate edge orientations.

Each polygonal schema Π defines a topological space $\Sigma(\Pi)$ by identifying pairs of sides with matching labels and orientations. The labels indicate which sides should be identified; the orientations indicate *how* the sides are to be identified. Each pair of identified sides becomes a single path in $\Sigma(\Pi)$, which we call an ***edge***. The identification of sides in Π induces an identification of corners. Thus, several corners may be mapped to the same point in $\Sigma(\Pi)$, which we call a ***vertex***. These vertices and edges define a graph $G(\Pi)$ embedded in $\Sigma(\Pi)$.

We can encode any polygonal schema by listing the labels and orientations of sides in cyclic order around each polygon. Whenever we traverse a side along its orientation, we record its label; when we traverse an edge against its orientation, we record its label with a bar over it. Thus, in the example schema on the previous page, traversing each polygon counterclockwise, starting from its bottom left corner, we obtain the signature $(\mathtt{a}\bar{\mathtt{b}}\mathtt{c}\bar{\mathtt{a}}\mathtt{db})(\mathtt{cde}\bar{\mathtt{e}})$. Of course, this is not the only possible encoding of this polygonal schema—we could traverse the square first, or start at a different side of each polygon, or reverse the traversal direction, or even permute the labels and/or reorient some sides. All such encodings and schemata are considered equivalent.

Alternately, we can record only the sequence of *edge* labels, and then separately record whether each edge is traversed in both directions or only in one direction. Thus, for the example schema can be encoded by the ***double permutation*** (abcadb)(cdee) and the function $\langle \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e} \rangle \mapsto \langle 2, 2, 1, 1, 2 \rangle$. If we regard every edge e as having two distinct ***shores***, called for example *left*(e) and *right*(e), then the double-permutation is actually a standard permutation of the shores. Let me emphasize that the names '*left*' and '*right*' have no geometric meaning whatsoever, at least for the moment.

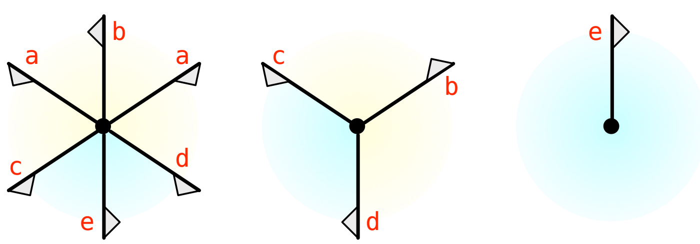
The idea of studying surfaces by cutting them into simply connected pieces dates back at least to Riemann [12]. Polygonal schemata and their higher-dimensional generalizations were developed by Poincaré as a combinatorial representation of abstarct topological spaces. Polygonal schemata were also used by Heffter [8] to encode cellular embeddings (see below) of the complete graph on orientable surfaces, in one of the earliest attack on Heawood's coloring conjecture [7]. In Dehn and Heegaard's seminal encyclopedia article [2], which established the formal foundations of combinatorial topology for the first time, 2-manifolds are *defined* to be topological spaces that can be described by a polygonal schema.

5.2 Cellular Graph Embeddings

A graph is a topological space: A collection of intervals with endpoints identified. In particular, we allow loops and parallel edges.

An ***embedding*** of a graph G into a 2-manifold Σ is a continuous injective map from G to Σ. The image of any graph embedding is homeomorphic to the graph itself. Thus, at the risk of confusing the reader, we usually use the same symbol G to simultaneously represent the abstract graph, the embedding map, and the image of the embedding. The components of $\Sigma \setminus G$ are called the ***faces*** of the embedding. An embedding is ***cellular*** (or ***2-cell***) if every face of G is homeomorphic to an open disk, and therefore to the plane.

In the neighborhood of any vertex v, any embedding cyclically orders the edges incident to v. Conversely, any cellular embedding is encoded by these cyclic permutations. A ***signed rotation system*** is a double-permutation of the edges, obtained by recording the labels of edges around each vertex in some order, together with a function $\iota : E \to \{1, 2\}$ indicating whether the cyclic orders at each end of an edge cross that edge in the same direction (↺—↻ or ↻—↺) or in opposite directions (↺—↺ or ↻—↻).



The signed rotation system induced by the example polygonal schema. Colors indicate faces of the embedding.

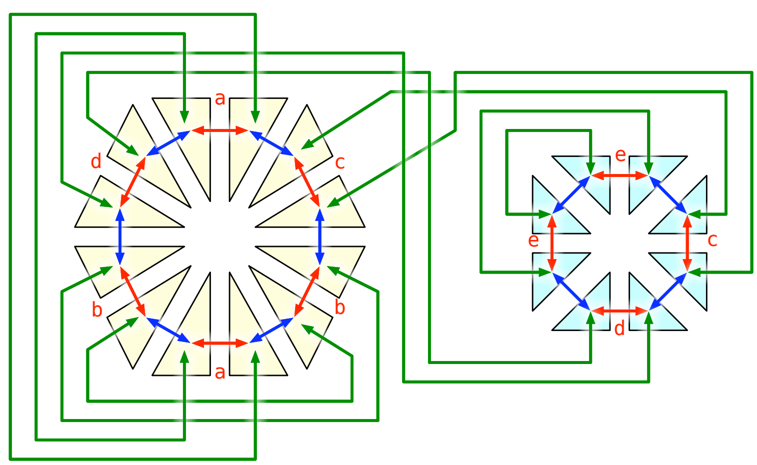
The study of graph embeddings on 2-manifolds was instigated by Heawood [7], who asked for the minimum number of colors required to properly color the vertices of a graph embedded on a surface (in the same paper where he pointed out a bug in Kempe's proof of the Four-Color Theorem). (Unsigned) rotation systems were first proposed by Edmonds [4] in 1960 to encode cellular embeddings of *simple* graphs, that is, without loops or parallel edges, on *orientable* surfaces; see also Youngs [14]. Youngs [15] added 'signs', thereby extending rotation systems to embeddings of simple graphs on non-orientable surfaces. Gross and Alpert [6] refined rotation systems further to support arbitrary multigraphs (with loops and parallel edges); they called the embedding induced by a rotation system the *Heffter-Edmonds imbedding*.

5.3 Barycentric Subdivision

The resemblance between signed rotation systems and polygonal schema encodings is not a coincidence; the two structures are in fact dual to each other. To my knowledge, Gross and Alpert [6] were the first to recognize this equivalence in print, almost fifteen years after Edmonds proposed rotation systems for the first time.

The ***barycentric subdivision*** of a polygonal schema Π, historically but inexplicably denoted $\boldsymbol{Sd(\Pi)}$, is obtained by subdividing each of the faces into triangles, which I will call ***flags***. The vertices of each flag are the centroid of a face f, the midpoint of a side e of that face, and an endpoint v of that side. Thus, any face with k sides is subdivided into $2k$ flags. Altogether, the barycentric subdivision of a polygonal schema with m edges (that is, $2m$ sides) contains $4m$ flags.

We call each vertex of $Sd(\Pi)$ an ***a-vertex*** (*apex*) if it is a vertex of Π, a ***b-vertex*** (*boundary*) if it is a midpoint of an edge of Π, and a ***c-vertex*** if it is the *centroid* of a face of Π. Similarly, we call an edge of $Sd(\Pi)$ an ***a-edge***, ***b-edge***, or ***c-edge*** depending on the type of the opposing vertex in either of the flags that contain it. Finally, for any flag F, we define $\boldsymbol{a(F)}$ to be the other flag sharing the same a-edge as F, and define $\boldsymbol{b(F)}$ and $\boldsymbol{c(F)}$ similarly. We can view the functions a, b, and c either as permutations on the set of all flags in $Sd(\Pi)$, or as the edges of a 3-regular graph whose vertices are the flags, as convenient.



The barycentric subdivision $Sd(\Pi)$ of the example polygonal schema Π.
Colored arrows indicate the three matchings: red $= a$, blue $= b$, green $= c$.

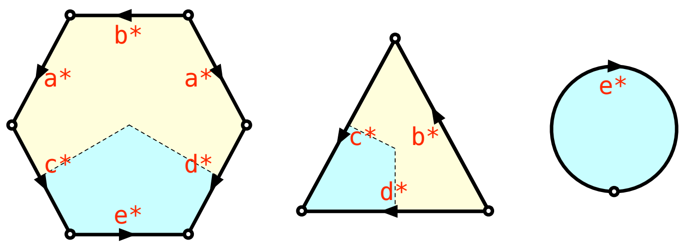
Also, by cutting the faces into their constituent flags, we can also view the barycentric subdivision of any polygonal schema as a finer polygonal schema for the same space. In particular, the vertices, edge midpoints, and face centroids in Π are all vertices in $Sd(\Pi)$.

Barycentric subdivisions are examples of (pure two-dimensional) *simplicial complexes*, which we will consider in much greater generality later in the course; roughly speaking, a simplicial complex is a set of simplices (points, segments, triangles, tetrahedra, etc.) glue together along common faces (of any dimension, and not necessarily in pairs).

5.4 Dual Schemata

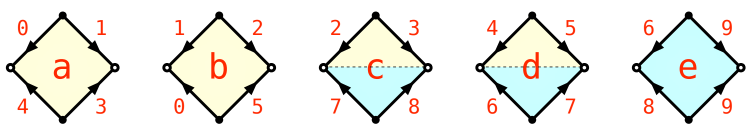
We can recover the original polygonal schema Π from its barycentric subdivision $Sd(\Pi)$ by gluing every flag f to the flags $a(f)$ and $b(f)$ along their common edges. The a and b links partition the set of flags into cycles; each cycle has a common centroid, and gluing the flags in that cycle recovers the face. The Jordan-Schönflies theorem implies that each face is a disk; the boundary of each face is covered with c-edges.

If instead we glue every flag f in $Sd(\Pi)$ to the flags $a(f)$ and $c(f)$ along their common edges, we obtain the ***dual schema*** $\boldsymbol{\Pi^*}$. Again, the a and c links partition the set of flags into several cycles, each with a common *vertex*. Gluing the flags in each cycle creates a disk with the vertex v in its interior (up to homeomorphism, at its center). This disk is the ***dual face*** $\boldsymbol{v^*}$. Note that a dual face may have only one or two sides; treating any such monogon or bigon as a geometric disk causes no problems. The centroid of each face f of Π becomes a ***dual vertex*** $\boldsymbol{f^*}$ in Π^*. For each edge e of Π, there is a corresponding ***dual edge*** $\boldsymbol{e^*}$ in Π^* with the same midpoint. The orientations of the sides of the dual faces can be determined by the b-matching between flags.



The dual schema Π^*; colors indicate faces of Π.

Finally, if instead we glue every flag f in $Sd(\Pi)$ to the flags $b(f)$ and $c(f)$ along their common edges, we obtain the ***diamond schema*** $\boldsymbol{\Pi^\diamond}$. The b and c links partition the flags into cycles, one for each edge e of Π. Gluing the flags in each cycle creates a quadrilateral with the midpoint of e in its interior (up to homeomorphism, at its center). This disk is a face of Π^*, denoted $e^\diamond$. Each vertex v of Π is also a vertex of $\Pi^\diamond$, as is each dual vertex f^* of Π^*. (Note that each side of a diamond face is a single flag edge, not two flag edges as in Π and Π^*.)



The diamond schema $\Pi^\diamond$; colors indicate faces of Π.

The barycentric subdivisions of any schema Π and its dual Π^* are combinatorially isomorphic, except for exchanging b's and c's. Thus, the dual schema of Π^* is combinatorially isomorphic to the original schema Π. The spaces $\Sigma(\Pi)$, $\Sigma(\Pi^*)$, and $\Sigma(\Pi^\diamond)$ are all clearly homeomorphic to $\Sigma(Sd(\Pi))$, and therefore homeomorphic to each other.

The relationship between primal and dual schemata also reveals the duality between schemata and rotation systems: Any polygonal schema Π is *combinatorially isomorphic* to the rotation system of the dual graph $G(\Pi^*)$. Specifically, the cyclic order of directed edges around any face f of Π *is* the cyclic order of sided edges around the dual vertex f^*.

5.5 Polygonal Schemas = 2-Manifolds

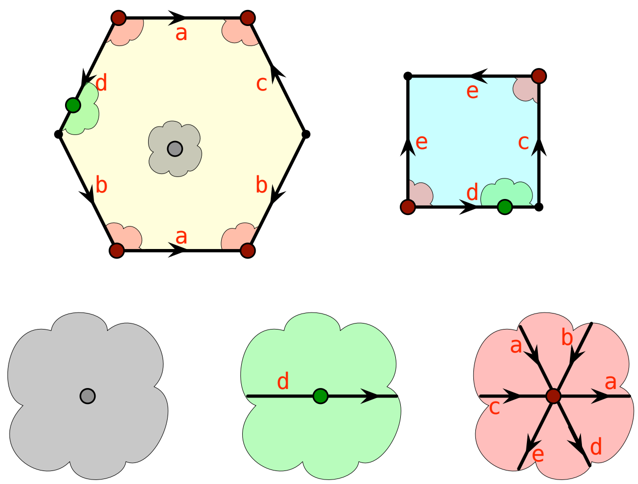
Now we can finally verify that polygonal schemata really describe the spaces we care about.

Lemma 5.1. *For any polygonal schema Π, the space $\Sigma(\Pi)$ is a compact 2-manifold.*

Proof: The ***star*** of any vertex x of $Sd(\Pi)$ is the set of all *open* faces of $Sd(\Pi)$ for which x is a vertex. The complex $Sd(\Pi)$ has three types of vertices. The star of the centroid of face f is the interior of f. The star of the midpoint of edge e is the interior of the diamond face $e^\diamond$, which contains the interior of e. Finally, the star of any vertex v of Π is the interior of the dual face v^*, which obviously contains the point v. Each of these subsets of $Sd(\Pi)$ is homeomorphic to an open disk, and therefore to the plane.

Consider an arbitrary point p in Σ; this point lies either in the interior of a face f of Π, in the interior of an edge e of Π, or at a vertex v of Π. Thus, p lies either in the interior of a face f, in the interior of a diamond face $e^\diamond$, or the interior of a dual face v^*. In each case, p lies in an open subset of Σ homeomorphic to the plane.

Finally, compactness of $\Sigma(\Pi)$ follows from the compactness of each face of Π. $\square$



Neighborhoods of three points in the example schema.

The converse of this lemma is quite a bit harder to prove. Until the early 20th century, it was simply *assumed* that any 2-manifold could be triangulated, that is, described by a simplicial complex.[1] The following lemma was first proved independently by Kerékjártó [9] and Radó [11] in the early 1920s. For shorter, more modern treatments, see Doyle and Moran [3] or Thomassen [13]. Here I will only give a brief sketch of a proof, with a huge chunk of the work unashamedly brushed under the rug.

Lemma 5.2 (Kerékjártó-Radó). *Any compact, connected 2-manifold can be described by a polygonal schema.*

Proof (sketch): Let Σ be a compact, connected 2-manifold. Because Σ is compact, it can be covered by a finite number of closed sets $D_1, D_2, \ldots, D_n$, each homeomorphic to a disk. We can assume without loss of generality that this collection of disks is minimal—no disk lies in the union of any other. ***With lots of grind***,[2] we can assume that for any i and j, the intersection $\partial D_i \cap \partial D_j$ consists of a finite number of points. It follows that $\Sigma \setminus \bigcup_{i=1}^n \partial D_i$ has a finite number of components. Each of these components is homeomorphic to a disk; otherwise, some points on Σ would not be covered. These components are the faces of a polygonal schema. The intersection points between boundary curves ∂D_i are the vertices, and arcs of boundary curves between vertices are the edges. $\square$

[1] In fact, it was assumed that *every* topological space can be triangulated. This assumption was proved true for 3-manifolds by Moise [10] and Bing [1] in the 1950s, but proved false(!) for 4-manifolds (and higher) by Freedman [5] in the 1980s.

[2] This is not the traditional meaning of 'WLOG', but it is often more accurate.

5.6 Orientability

Theorem 5.3. *Let* Π *be a polygonal schema of complexity* n*. We can determine in* $O(n)$ *time whether* $\Sigma(\Pi)$ *is orientable.*

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