Wagner did indeed discuss this problem in the 1960s with his then students, Halin and Mader, and it is not unthinkable that one of them conjectured a positive solution. Wagner himself always insisted that he did not—even after the graph minor theorem had been proved.

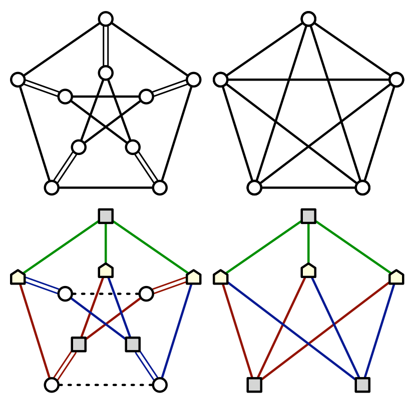
— Reinhard Diestel, *Graph Theory*, 3rd edition (2005)

Unfortunately, for any instance $G = (V, E)$ that one could fit into the known universe, one would easily prefer $|V|^{70}$ to even constant time, if that constant had to be one of Robertson and Seymour's.

— David Johnson, "The NP-completeness Column" (1987)

12 Graph Minors

A ***minor*** of a graph G is a graph obtained from G by contracting edges, deleting edges, and deleting isolated vertices; a ***proper minor*** of G is any minor other than G itself. For example, the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are both minors of the infamous Peterson graph:



Both K_5 and $K_{3,3}$ are minors of the Peterson graph.
Doubled edges are contracted; dashed edges are deleted.

A classical theorem of Kuratowski [21] states that a graph is planar if and only if it does not contain a ***subdivision*** of K_5 or $K_{3,3}$ as a subgraph. Kuratowski's theorem was refined by Wagner in his 1935 PhD thesis:

Theorem 12.1 (Wagner [43]). *A graph G is planar if and only if K_5 and $K_{3,3}$ are not minors of G.*

Wagner's thesis continued with a characterization of all graphs that do not have K_5 (but may have $K_{3,3}$) as a minor [42]. Wagner's work led to a more general study of families of graphs with ***forbidden minors***. A graph H is a forbidden minor for a set $\mathcal{F}$ of graphs if H is not a minor of any graph in $\mathcal{F}$. A forbidden minor H of $\mathcal{F}$ is ***minimal*** if no proper minor of H is also a forbidden minor.

It's quite easy to see that any family $\mathcal{F}$ of graphs with at least one forbidden minor is ***minor-closed***: Every minor of a graph in $\mathcal{F}$ is also in $\mathcal{F}$. Conversely, if a minor-closed family of graphs excludes a graph H, it must also exclude any graph for which H is a minor. Thus, every minor-closed family of graphs, except the family of *all* graphs, has at least one forbidden minor.

In the mid-1980s, Neil Robertson and Paul Seymour announced a proof of one of the deepest theorems in combinatorics [25]; the details of their proof were published over the next two decades in a series of 21 papers totalling several hundred pages. According to Robertson and Seymour, this theorem was conjectured by Wagner as early as the 1930s, although his conjecture did not appear in print until many decades later [44].

The Graph Minor Theorem (Robertson and Seymour [29]). *In any infinite set of graphs, at least one graph is a proper minor of another.*

Even a brief sketch of the proof of the Graph Minor Theorem is far beyond the scope of this class. Instead, I will confine myself to a list, with few proofs, of partial results of Robertson, Seymour, and others that are interesting in their own right.

12.1 Minor-Closed Families

We have already seen two examples of minor-closed families of graphs. First, for any 2-manifold Σ, the family of graphs that can be embedded (not necessarily cellularly) on Σ is minor closed. If a graph has an embedding on Σ, then deleting and contracting edges cannot introduce crossings into the embedding. In particular, the set of all planar graphs is minor-closed. Second, for any integer k, the graphs with treewidth at most k define a minor-closed family. Any tree decomposition for a graph G is also a valid tree decomposition for any subgraph of G; if we contract any edge uv to a new vertex w, we can repair the tree decomposition by replacing any occurrences of u or v (or both) in any $X(i)$ with w. In particular, the set of forests is minor-closed.

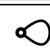
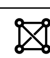
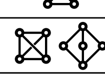
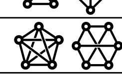
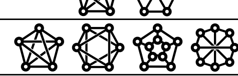
Two more interesting minor-closed families involve embeddings of graphs in $\mathbb{R}^3$. An embedding of a graph G is ***knotless*** if no cycle in G is knotted, and ***linkless*** if not to cycles in G are linked. (I'll define knots and links more formally later in the semester, but they mean what you think they do.) Deleting an edge cannot introduce knots or links into an embedding, moreover, we can contract edges without introducing knots or links. Thus, the sets of ***linklessly embeddable*** and ***knotlessly embeddable*** graphs are minor-closed.

The following theorem is an immediate corollary of the Graph Minor Theorem:

Theorem 12.2. *A family of graphs is minor-closed if and only if it has a finite number of minimal forbidden minors.*

Thus, ***all*** minor-closed families of graphs have Kuratowski-Wagner-type theorems associated with them; the set of minimal forbidden minors is sometimes called an ***obstruction set*** for the graph family. Special cases of this theorem were proved independently by Vollmerhaus [40, 41] and Bodendiek and Wagner [7] for graphs embeddable on fixed orientable surfaces, and by Archdeacon and Huneke [4] for graphs embeddable on fixed non-orientable surfaces.

Unfortunately, the proof of the Graph Minor Theorem is nonconstructive, and therefore does not yield an explicit list of forbidden minors for any minor-closed family. Explicit obstruction sets are known only for a few special cases; here is an incomplete list:

Family	Obstruction set	
treewidth 1 (forests)		
treewidth 2		
outerplanar		
planar		[43]
treewidth 3		[6, 34]
linklessly embeddable	7 graphs	[32, 33, 31]
projective-planar	35 graphs	[16, 3]

Chambers and Myrvold have found 16,629 distinct forbidden minors for graphs embeddable on the torus, but their list is is probably not exhaustive [15]. Adler, Grohe, and Kreutzer [1] described (but did not run) an algorithm to compute the obstruction set for certain graph classes, including graphs of any fixed genus and/or treewidth. A fully constructive proof for the general case is impossible: Fellows and Langston [14] proved that there is no algorithm to compute the obstruction set of an arbitrary minor-closed family $\mathcal{F}$, represented as a Turing machine that enumerates the elements of $\mathcal{F}$.

However, there is an algorithm to determine whether one graph is a minor of another.

Theorem 12.3 (Robertson and Seymour [27]). *For any fixed graph H, there is an algorithm to determine whether a given n-node graph has H a a minor in $O(n^3)$ time.*[1]

Together with the Graph Minor Theorem, this immediately implies the *existence* of an $O(n^3)$-time algorithm to test membership in any fixed minor-closed family. Again, this theorem is non-constructive; its proof does not yield an explicit description of the algorithm, only knowledge of its existence. Moreover, the constant hidden in the big-Oh bound is a truly *enormous* function of the number of vertices in H.[2] Simply exponential dependence on H is unavoidable; determining whether a given graph has a path, cycle, or clique minor of a given size is NP-complete [5, 13].

Much faster algorithms are known for a few special cases. For any fixed k, there is an algorithm to check whether a graph has treewidth at most k in $O(n)$ time [8]. Similarly, for any fixed g, there is an algorithm to determine whether a graph can be embedded in a surface of genus g in $O(n)$ time [22, 18]. Both algorithms either build the required structure (tree decomposition or rotation system) or find a forbidden minor in the graph. Both running times hide an exponential dependence on the fixed parameter (k or g); if those parameters are allowed to vary, the problems become NP-hard [39]. For most minor-closed families, however, no explicit membership algorithm is known; in particular, no explicit algorithm is known to test whether a graph has a knotless embedding.

12.2 Properties of Minor-Closed Families

Minor-closed families have many of the same combinatorial properties as planar graphs and more general surface graphs. Thus, many algorithms designed for surface graphs extend (almost) automatically to graphs in any minor-closed family.

Theorem 12.4 (Kostochka [19, 20]; Thomason [37, 38]). *For every fixed graph H, any n-vertex H-minor-free graph has $O(n)$ edges. In particular, any n-vertex K_k-minor-free graph has $O(nk\sqrt{\log k})$ edges; this bound is tight on average for random K_k-minor-free graphs.*

Theorem 12.5 (Alon, Seymour, and Thomas [2]; Plotkin, Rao, and Smith [23]). *For every fixed graph H, any n-vertex H-minor-free graph has treewidth $O(\sqrt{n})$. In particular, any n-vertex K_k-minor-free graph has treewidth $O(k\sqrt{n}\sqrt{\min\{k, \log n\}})$.*

Unlike all our earlier separator bounds, there is no matching lower bound in this setting; the correct treewidth bound is conjectured to be $O(k\sqrt{n})$.

Theorem 12.6 (DeVos *et al.* [11]; Demaine, Hajiaghayi, and Kawarabayashi [10]). *For every fixed graph H and any integer $k \geq 2$, any n-vertex H-minor-free graph is the union of k subgraphs, each with treewidth $O(k)$.*

[1] Reed, Kawarabayashi, and Li have announced a minor-testing algorithm that runs in $O(n \log n)$ time [24], but their algorithm has not yet been published.

[2] Johnson [17] estimated that the hidden constant is "somewhat larger" than $2 \Uparrow (2 \Uparrow (2 \Uparrow (h/2)) + 3)$, where $2 \Uparrow t$ denotes an exponential tower of t 2s ($2 \Uparrow 0 = 1$ and $2 \Uparrow t = 2^{2 \Uparrow (t-1)}$) and h is the number of vertices in H.

12.3 Grids and Treewidth

The ***r* × *r* grid** is a graph $G(V, E)$, where $V = \{1, 2, \ldots, r\} \times \{1, 2, \ldots, r\}$ and two pairs (i, j) and (i', j') are connected by an edge if and only if $|i - i'| + |j - j'| = 1$. Grids are the canonical example of Grids are the canonical example of graphs with large treewidth; it it not hard to prove that the $r \times r$ grid has treewidth exactly r. As part of the proof of the Graph Minor Theorem, Robertson and Seymour proved that in a sense, grids are the *only* example of a graph with large treewidth.

Lemma 12.7 (Robertson and Seymour [26]). *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer r, every graph of treewidth at least $f(r)$ has an $r \times r$ grid minor.*

The precise function f is not known; in fact, there is an exponential gap between the best known bounds $2^{O(r^5)}$ and $\Omega(r^2 \log r)$ [30]. A relatively simple proof of Lemma 12.7 is given by Diestel *et al.* [12].

Theorem 12.8. *A minor-closed family of graphs has bounded treewidth if and only if it excludes at least one planar graph.*

Proof: One direction is trivial: Any set of graphs that contains all planar graphs contains all grids, and therefore contains graphs of arbitrarily large treewidth.

Now suppose $\mathcal{F}$ is a minor closed family that excludes some r-vertex planar graph. Any r-vertex planar graph can be embedded as a minor in an $O(r) \times O(r)$ grid [35, 36]. Thus, $\mathcal{F}$ also excludes some $O(r) \times O(r)$ grid, which implies by Lemma 12.7 that every graph in $\mathcal{F}$ has treewidth less than some constant $f(O(r))$. □

This theorem, together with Theorem 12.5, implies that the worst-case treewidth of an n-vertex graph in any minor closed family $\mathcal{F}$ is either $\Theta(\sqrt{n})$ (if $\mathcal{F}$ includes all planar graph) or $\Theta(1)$ (if $\mathcal{F}$ excludes some planar graph); there are no other possibilities.

Demaine and Hajiaghayi recently proved a stronger version of Lemma 12.7 for graphs that forbid a *fixed* minor.

Lemma 12.9 (Demaine and Hajiaghayi [9]). *For any fixed graph H, every H-minor-free graph of treewidth w has an $\Omega(w) \times \Omega(w)$ grid as a minor.*

12.4 Decomposition Theorem

Finally, Robertson and Seymour describe a canonical structure for all graphs in any minor-closed family. Roughly speaking, any graph from any minor-closed family can be decomposed hierarchically into graphs with small genus plus a small amount of extra 'noise'.

A ***clique-sum*** of two graphs G and H is any graph obtained by identifying a clique in G with a clique of the same size in H, and then possibly removing some edges in the resulting shared clique. A ***k-clique-sum*** is a clique-sum in which the identified cliques have at most k vertices. We write $G \oplus H$ to describe any clique sum of graphs G and H, and let $G_1 \oplus G_2 \oplus \cdots \oplus G_r = (G_1 \oplus G_2 \oplus \cdots \oplus G_{r-1}) \oplus G_r$. Every k-tree is a clique-sum of $(k+1)$-cliques (with no edge deletions). Every graph with treewidth k is a k-clique-sum of graphs with at most $k+1$ vertices.

A graph H is a ***k-apex graph*** of a graph G if $G = H \setminus A$ for some subset A of at most k vertices, called ***apices***.

Finally, a ***vortex*** is a graph with small treewidth that is glued into a face of an embedded graph in a particular way; in fact, we need a stronger version of treewidth to define vortices properly. A ***path decomposition*** of a graph is a tree decomposition (T, X) where T is a path. Equivalently, a path decomposition of a graph $G = (V, E)$ is a *sequence* $X = \langle X_1, X_2, \ldots, X_r \rangle$ of subsets of V, where

- $\bigcup_i X_i = V$;
- For every edge $uv \in E$, we have $u, v \in X_i$ for some i;
- If $v \in X_i \cap X_k$ for some integer $i \le k$, then $v \in X_j$ for all $i \le j \le k$.

The ***width*** of a path decomposition is one less than the size of the largest set X_i. The ***pathwidth*** of a graph is the minimum width of any path decomposition.

A graph G is ***cleanly k-almost-embeddable*** on a surface Σ if G can be written as the union of $k+1$ graphs $G_0 \cup G_1 \cup \cdots k$, satisfying the following conditions:

- G_0 has an embedding on Σ (which need not be cellular).
- The graphs $G_1, G_2, \ldots, G_k$ are pairwise disjoint; these subgraphs are the vortices of G.
- For each index $i \ge 1$, there is a disk D_i inside some face F_i of G_0, such that $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap D_i$. Moreover, the disks D_i are pairwise disjoint.
- For each index $i \ge 1$, the subgraph G_i has pathwidth less than k. Moreover, G_i has a path decomposition $\langle X_1, X_2, \ldots, X_r \rangle$ of width less than k, such that $v_{ir_i} \in X_i$ for all i, where $v_{i1}, v_{i2}, \ldots, v_{ir_i}$ are the vertices of U_i indexed in cyclic order around the face F_i.

A graph G is ***k-almost-embeddable*** in a surface Σ if H is a k-apex graph of a graph that is cleanly k-almost-embeddable in Σ.

Theorem 12.10 (Roberston and Seymour [28]). *For any graph H, there is an integer $k = k(H)$ such that any H-minor-free graph is a k-clique sum of a finite number of k-apex graphs of cleanly k-almost-embeddable graphs on the orientable surface of genus k.*

Demaine, Hajiaghayi, and Kawarabayashi [10] describe an algorithm to compute a decomposition in this canonical form in polynomial time, for any fixed forbidden minor H.

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