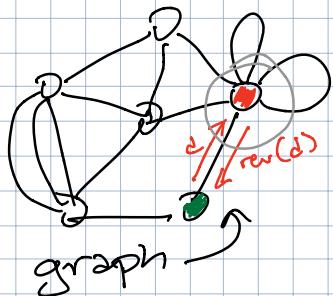


## Planar Graphs

$$G = (V, E)$$

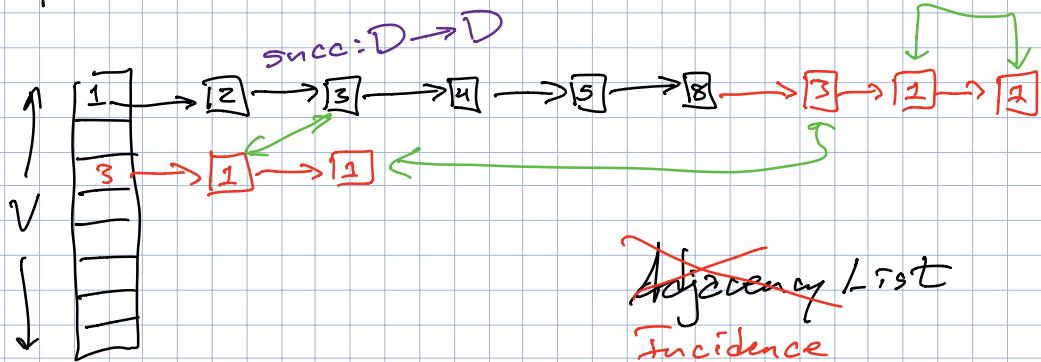
$V$  - finite set of vertices

$E$  - pairs of vertices - edges



↑  
simple graph

graph



~~Adjacency List  
Incidence~~

$$\text{Graph} = (V, D, \text{rev}, \text{head})$$

$(D, \text{rev}, \text{succ})$

$V$  = vertices

edge = orbit rev  
vertex = orbit succ

$D$  = darts even #

$\text{rev}: D \rightarrow D$

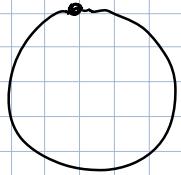
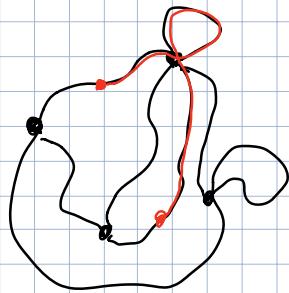
involution:  $\text{rev}(\text{rev}(d)) = d \neq \text{rev}(d)$

$\text{head}: D \rightarrow V$

Edge  $e = \{d, \text{rev}(d)\}$

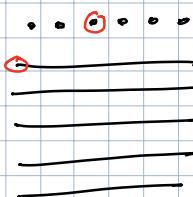
$\deg(v) = \# \text{head}^{-1}(v)$

## Topological graphs



$V^T$  = distinct points

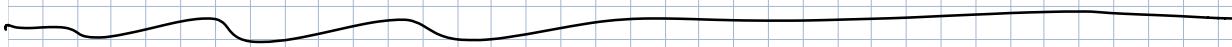
$E^T$  = distinct real intervals



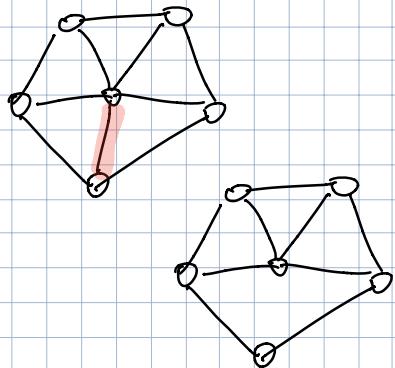
$$G = (V^T \cup E^T) / \sim$$

$$e^T(0) \sim v^T \text{ if } e = uv$$

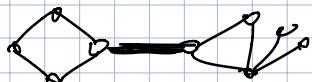
$$e^T(1) \sim u^T$$



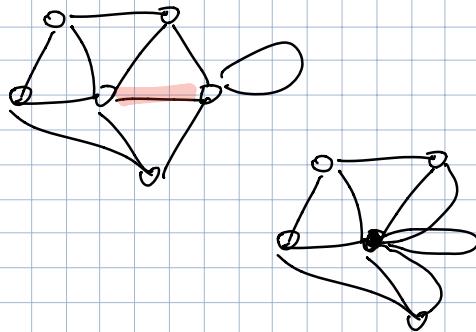
Deletion



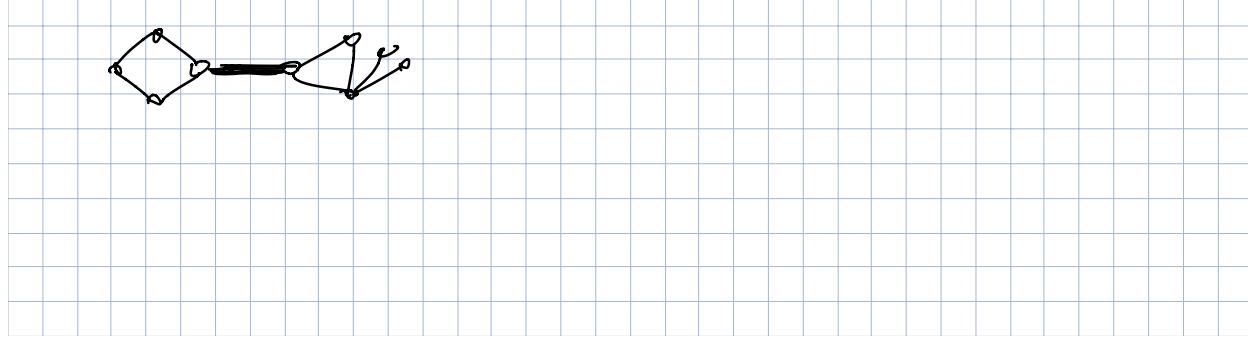
If  $e$  delete is not  
a bridge,  $G/e$   
is connected

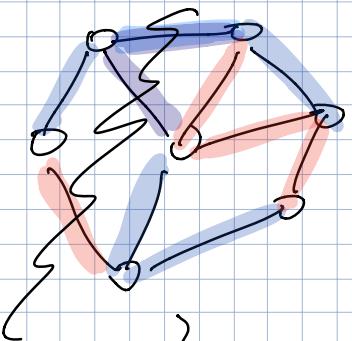


Contraction



If  $e$  is not a loop  
then  $G/e$   
makes sense





For all edges  $e$

if  $e$  is a loop

delete  $e$

else

contract  $e$

Contracted edges define

a spanning tree



Color edges arbitrarily

- every cycle has  $\geq 1$  red edge

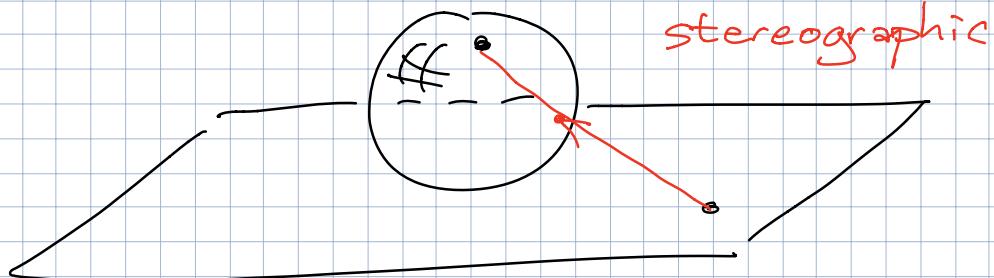
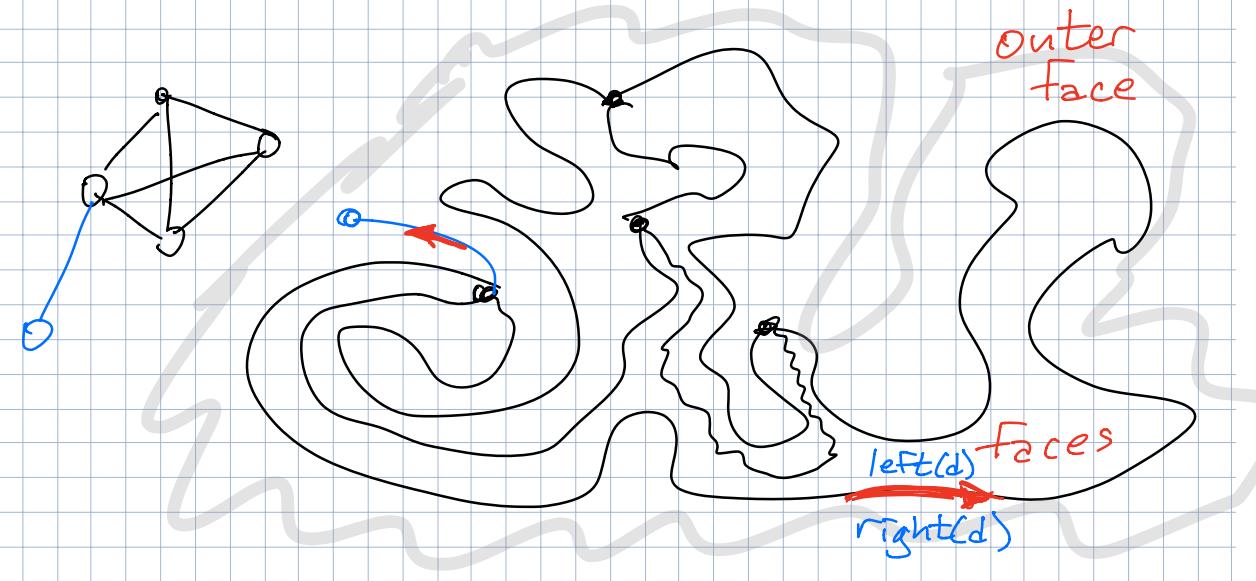
- every cut has  $\geq 1$  blue edge

$T_{\text{blue}} = \text{spanning tree}$ .

## Planar graph

$G$  is planar if

there is an embedding  $\phi: G^T \hookrightarrow \mathbb{R}^2$

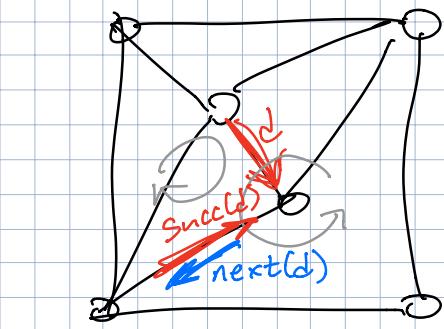


$$\deg(F) = \# \underset{\text{right}}{\text{left}}^{-1}(F)$$

$\text{next}: D \rightarrow D$

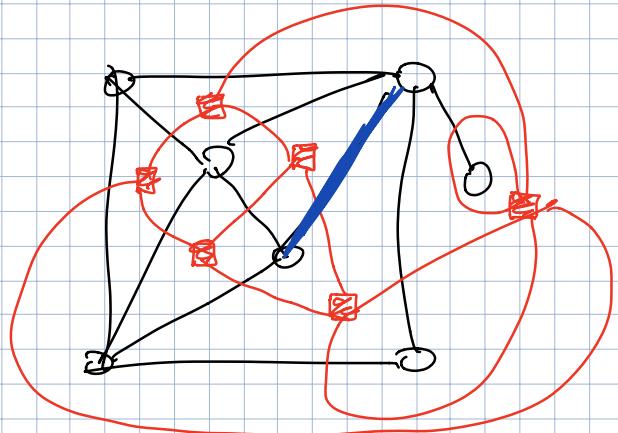
encodes darts with same  
~~left~~ shore in cyclic order  
~~right~~ clockwise

$\text{succ}: D \rightarrow D$



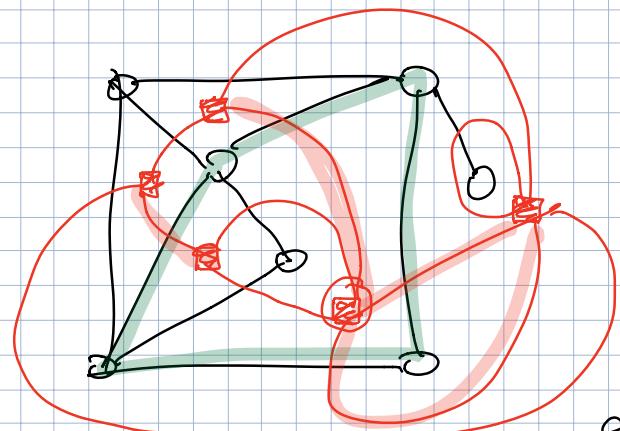
$$\text{succ} = \text{rev} \circ \text{next}$$

$$\text{next} = \text{rev} \circ \text{succ}$$



$G$  planar map

$G^*$  dual map



deletion = contraction\*

cycle = cut\*  
[Whitney]

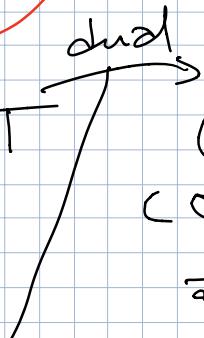
spanning tree  $T$

acyclic

connected

$(E \setminus T)^*$

connected  
acyclic

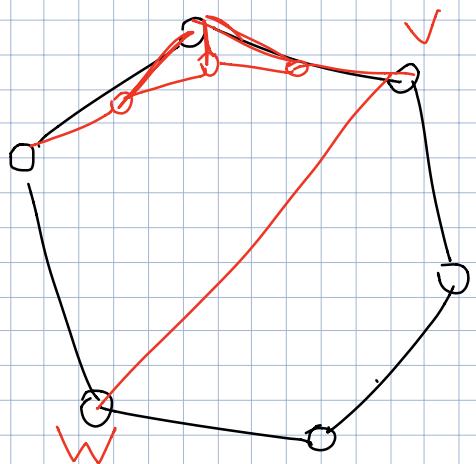


## Straight-line embedding



Theorem

Every simple planar graph has a straight emb.



WLOG

every face is a  $\triangle$   
except outer simple

