

## Homotopy

Definition: continuous deformation

A homotopy from  $\alpha: S^1 \rightarrow \mathbb{R}^2$  to  $\beta: S^1 \rightarrow \mathbb{R}^2$

is a continuous function  $h: S^1 \times [0,1] \rightarrow \mathbb{R}^2$  s.t.

$$h(\theta, 0) = \alpha(\theta) \quad h(\theta, 1) = \beta(\theta)$$

Intuitively, second parameter is "time"

There is a homotopy between any two planar curves:

$$h(\theta, t) = (1-t) \cdot \alpha(\theta) + t \cdot \beta(\theta)$$

But linear interpolation isn't necessarily "nice"

## Combinatorial Homotopy — Depends on curve representation

Polygons: Triangle moves

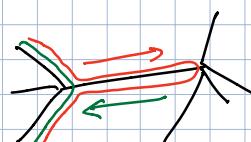
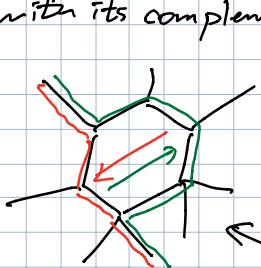


or Vertex moves



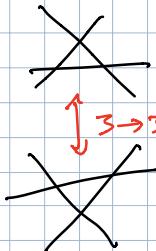
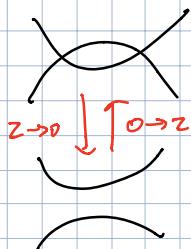
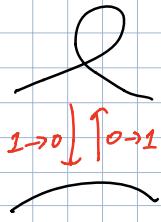
Graph walks: Face moves and edge moves

replace part of face boundary  
with its complement



Suffices to allow adding/removing  
entire face boundaries

Generic curves: "1-Homotopy moves" [Steinitz, Alexander, Briggs,  
Reidemeister, Tits, ...]



+ isotopy (free)

In all models, an arbitrary homotopy between nice curves is equivalent to a finite sequence of moves.

Formally, homotopic — There is a function

$$H: S^1 \times [0,1] \times [0,1] \rightarrow \mathbb{R}^2$$

$$\text{s.t. } H(\cdot, \cdot, 0) = h \text{ and } H(\cdot, \cdot, 1) = h'$$

Given

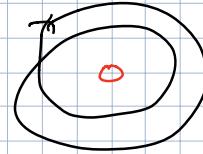
Finite # moves

Except in the graph model

Moreover,  $\|H(\theta, t, u) - h(\theta, t)\| < \varepsilon$   
 for all  $\theta, t, u$   
 for any desired  $\varepsilon > 0$ .

"Simplicial approximation theorem"

Obstacles Suppose we want our homotopy to avoid some point in the plane. WLOG  $o = (0,0)$ .



Now not every pair of curves is homotopic

$$h: S^1 \times [0,1] \rightarrow \mathbb{R}^2 \setminus o$$

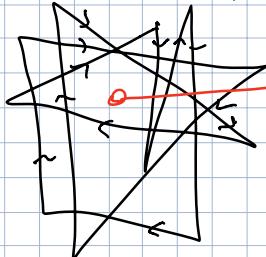
[Hopf 1935]

Lemma:  $\alpha$  and  $\beta$  are homotopic curves in  $\mathbb{R}^2 \setminus o$

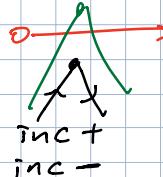
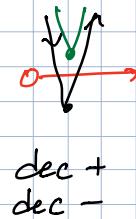
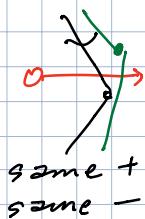
if and only if  $\text{wind}(\alpha, o) = \text{wind}(\beta, o)$

Proof:  $\Rightarrow$  (Project  $\alpha, \beta, h$  onto  $S^1$ , argue about degree)

" $\Rightarrow$ " Consider polygonal homotopy



$$\text{wind} = -1 + 1 - 1 - 1 - 1 = -3$$



$\Leftarrow$  (Linear interpolation of polar coords.) Exercise for polygons/generic  $\square$

We say that winding # is a homotopy invariant

Algorithm problem: How many moves do we need?  
Interesting even without obstacles.

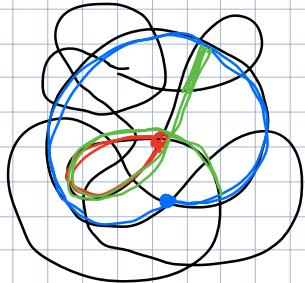
Polygons:  $O(n)$  [trivial] Graph walks:  $O(n+k)$   
graph  $\xrightarrow{\quad}$  walk

Homotopy moves:  $O(n^2)$  [Steinitz]

Proof: Fix your favorite curve  $\gamma$ , let  $n = \#$  vertices

If  $\gamma$  is simple, nothing to do, so assume  $n > 0$ .

Any vertex splits  $\gamma$  into smaller curves



By induction, some vertex splits off a simple subcurve

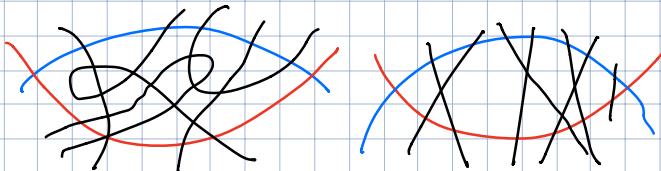
loop or butt

↓ if loop is empty, can be removed by 1  $\Rightarrow$   
↓ can't be empty (by definition/SC)

If not empty, some strand crosses.

— if strand is non-simple, recurse on simple subcurve with fewer faces

Simple strand creates a bigon.



**Lemma:** Every non-simple curve contains an empty loop or a minimal bigon.

An inclusion-minimal / irreducible bigon:

All crossing strands are simple  
Any pair crosses at most once

We can remove a minimal bigon with  $3 \rightarrow 3$ 's and one  $2 \rightarrow 0$



Euler  $\Rightarrow$  every min bigon has a triangular face on its boundary

#moves = #faces of bigon  $\leq$  #faces of  $\gamma = n - 2$  [Euler]

So we can either remove 1 vertex with 1 move  
or 2 vertices with  $\leq n - 2$  moves

IH  $\Rightarrow \square$

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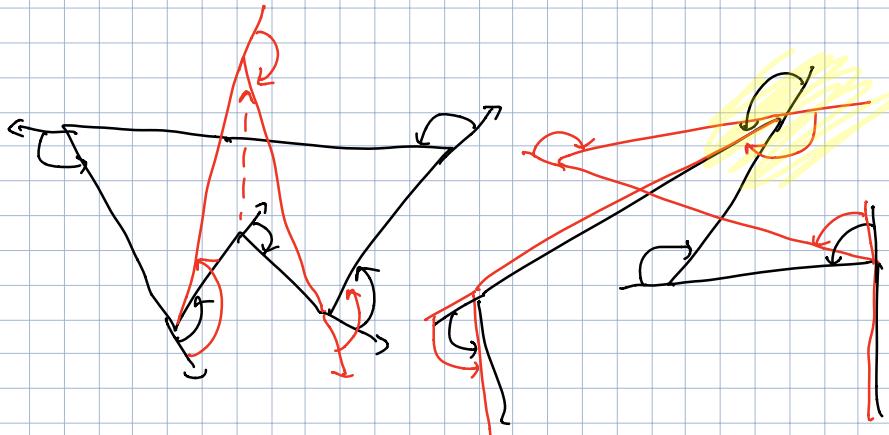
This is not optimal.  $\Theta(n^{7/2})$  [CE 17] / It is optimal in  $\mathbb{R}^2 \setminus O$ ! [CE 18]

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Regular homotopy / rotation #

Bradyardine [1350]

Meister [1777] — vertex moves don't change sum of external angles  
except when they collapse corners



A regular polygonal homotopy avoids spurs/spikes/whiskers/ $0^\circ$  angles

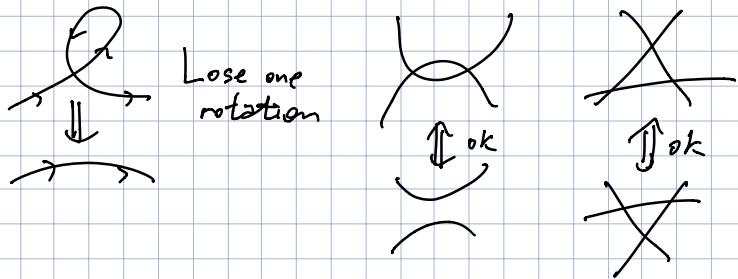
Rotation# =  $\sum$  ext angles is a regular homotopy invariant.

Lemma 2: Two polygons are reg homotopic iff same rotation#

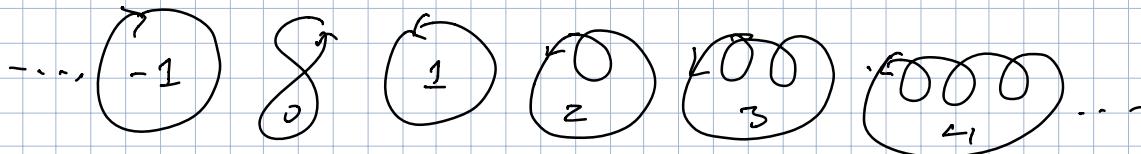
Whitney-Gruenbaum Theorem: Arbitrary curves  
are reg. homotopic  $\Leftrightarrow$  equal rotation #

"regular" = continuous non-zero tangents

Generic curves: Allow only  $2 \rightarrow 0$  and  $3 \rightarrow 3$  moves

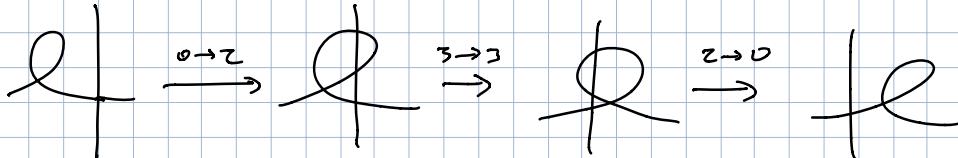


Theorem: Any curve can be transformed into a canonical curve  
with the same rotation number using  $O(n^2)$   
reg. homotopy moves



Proof: By Steinitz, there is an empty loop or bigon

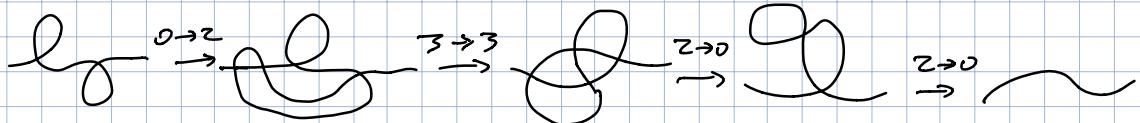
Loop: slide to neighborhood of basepoint and ignore (for now)



Bigon: remove 2 vertices in  $O(n)$  moves.

Eventually reach circle with loops

Apply "Whitney trick" to  
cancel opposite loops



This is optimal. [Nowik 2014]

□

Curve invariants: [Arnold, Arcadi, Polyak, et al.]

$$\text{Defect} = -2 \sum_{x \in \gamma} \text{sgn}(x) \cdot \text{sgn}(y) \quad [\text{Shumakovitch}]$$

$$\text{Strangeness} = \underbrace{\left( \text{wind}(\gamma, \gamma(0))^2 - \frac{1}{4} \right)}_{\text{halfInt}} + \sum_x \underbrace{\text{sgn}(x) \text{wind}(\gamma, x)}_{\text{int}}$$

Lemma: Defect + strangeness are independent of basept + orientation

Lemma: Defect + strangeness change as follows:

$$\begin{array}{ll} 1 \rightarrow 0: & \text{Defect unchanged} \\ 2 \rightarrow 0: & \text{change by } -2, 0, 2 \\ 3 \rightarrow 3: & \text{change by } \pm 2 \end{array}$$

Strangeness changes by  $\text{wind}(x)$   
unchanged  
changes by  $\pm 1$

Lemma:  $\text{Defect}(\gamma) = 0$  ————— Canonical curves have  $\text{St} = 0$

Lemma: There are  $n$ -vertex curves with  $\text{defect} = \mathcal{O}(n^{3/2})$

