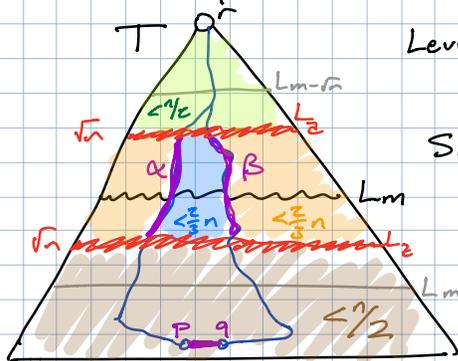




Suppose neither of these is small.  
 So  $\text{depth}(T) > \text{depth}(uv^*) \gg \sqrt{n}$



Levels  $L_{m-\sqrt{n}}$  through  $L_m$  contain  $n$  vertices  
 so  $|L_2| \leq \sqrt{n}$  for some  $m-\sqrt{n} \leq a < m$

Similarly  $|L_2| \leq n$  for some  $m < z \leq m+\sqrt{n}$

Let  $\alpha$  = segment of shortest path  
 from  $r$  to  $p$  between  $L_2$  and  $L_3$   
 $\beta$  = ..... to  $q$ .....

$|\alpha| \leq 2\sqrt{n}$   $|\beta| \leq 2\sqrt{n}$

$L_2 \cup L_3 \cup \alpha \cup \beta$  is a balanced separator of size  $\leq 6\sqrt{n}$

Cycle separators — [Miller<sup>86</sup>] — ... — [Nayyeri Har-Peled<sup>17</sup>]

Now suppose we want to separate the faces

Let  $T = \text{BFS tree}$  —  $\text{cycle}(T, pq)$  balanced separator

Let  $r = \text{lca}(p, q)$ , let  $T = \text{BFS tree}$  rooted at  $r$   
 s.t.  $\text{cycle}(T, pq) = \text{cycle}(T', pq)$

[For example, perturb edge wts and let  
 $T', T$  be unique shortest path trees

Wlog  $h := \text{dist}(r, p) \geq \text{dist}(r, q)$  and  
 $p$  is on outer face of  $G$ . (so  $T$  grows "outward" for first  $h$  levels)

Define level of any face  $F = \max$  distance from  $r$  to vertices of  $F$

For all  $i < h$ , define

$R_{\leq i} = \text{union of faces with level } \leq i$  — disk with holes

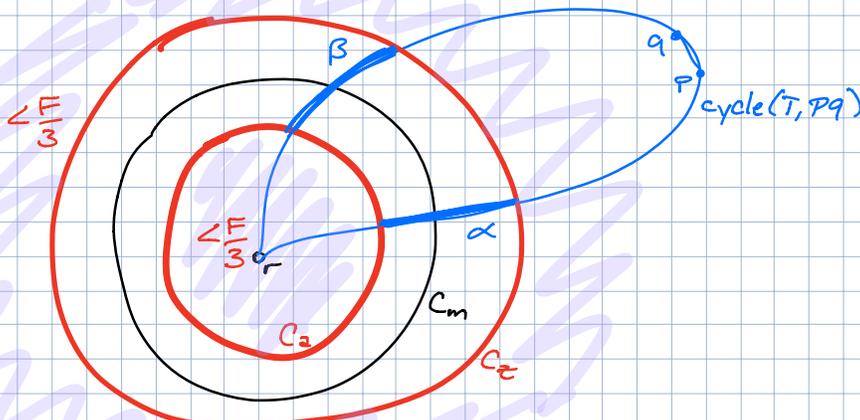
$C_i = \text{outer boundary of } R_{\leq i}$

- $\Rightarrow$  All vertices at distance  $i$  from  $r$
- $\Rightarrow$  Simple cycle
- $\Rightarrow$  disjoint from  $C_j$  for all  $i \neq j$ .
- $\Rightarrow$  intersects  $\text{cycle}(T, pq)$

Now assume  $h \gg \sqrt{n}$  (since o/w already done)

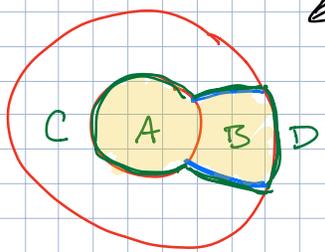
Let  $m = \max \{i \mid \leq \frac{1}{2} \text{ Faces inside } C_i\}$

At most  $\frac{1}{2}$  faces outside  $C_m$



$|C_a| < \sqrt{n}$  for some  $m - \sqrt{n} \leq a < m$   
 $|C_z| < \sqrt{n}$  for some  $m < z \leq m + \sqrt{n}$

← Here we have a balanced sep, but not a cycle sep.



Every region of this graph contains  $\leq \frac{2}{3}F$  Faces of  $G$ .

If any region contains  $> \frac{F}{3}$  Faces, done.

Otherwise  $\frac{F}{3} \leq |A \cup B| < \frac{2F}{3}$  and done.  $\square$

Easy to construct in  $O(n)$  time

with more effort: We can balance arbitrary\* weights on vertices, edges, and/or faces

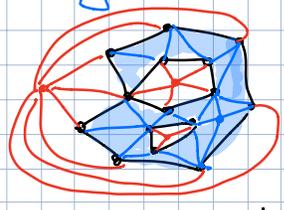
(\*as long as no single thing has huge weight)

r-division: Recursively partition into  $O(\frac{n}{r})$  pieces each with  $O(r)$  vertices and  $O(\sqrt{r})$  boundary vertices (shared w/other pieces)

Moreover, each piece is a disk with  $O(1)$  holes

Triangulate holes at each recursive call

total  $O(\frac{n}{r})$  bdr vertices



Alternately partition

- natural vertices = not wheel vertices
- boundary vertices
- holes = wheel vertices

[Klein Moses Sommer '13]

Naive:  $O(n \log(\frac{n}{r}))$  time

In fact we can build entire subdivision hierarchy in  $O(n)$  time with these properties [Goodrich 05] [KMS 15]

## Shortest paths

① Build nice  $r$ -division —  $O(n)$  time Assume  $s$  is a bdy vertex  $r$  to be determined

② For every piece, compute all boundary-to-boundary distances

$$O(1) \times \text{MSSP} = O(r \log r) \text{ per piece} \\ = O(n \log r) \text{ altogether}$$

③ Replace each piece with a clique over bdy vertices

Each clique has  $O(\sqrt{r})$  vertices  $\Rightarrow O(r)$  edges

Dense distance graph has  $O(\frac{n}{\sqrt{r}})$  vertices +  $O(n)$  edges

④ Compute shortest paths in DDG via Dijkstra

$$- O(E + V \log V) = O(n + \frac{n}{\sqrt{r}} \log n) \text{ time}$$

Now we know distances from  $s$  to every bdy vertex

⑤ Run Dijkstra within each piece

(extre edges from  $s$  with correct distances)

$$- O(\frac{n}{\sqrt{r}}) \times O(r \log r) = O(n \log r) \text{ time}$$

$$\text{Total time} = O(n \log r + \frac{n}{\sqrt{r}} \log n) = O(n \log \log n) \leftarrow$$

Balance by making two terms equal  $\sqrt{r} \log r = \log n$

$$\Rightarrow r = \left( \frac{\log n}{\log \log n} \right)^2$$

With a LOT more work, shortest paths in  $O(n)$  time

[Henzinger Klein  
Rao Subramanian 97]