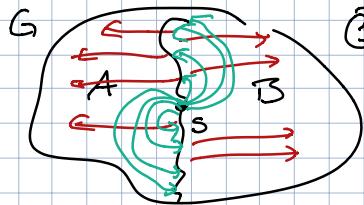


Shortest Paths with Neg. Lengths

Bellman-Ford: $O(VE) = O(n^2)$

[Lipton Rose Tarjan 79] - $O(n^{3/2})$ via separators



① Recursively compute $\text{dist}_A(s, A)$, $\text{dist}_B(s, B)$

Use these distances to reprice edges in A and B

$$w_A(u \rightarrow v) = \text{dist}_A(s, u) + w(u \rightarrow v) - \text{dist}_A(s, v)$$

Lemma: shortest wrt w_A = shortest wrt w

② For all $v \in S$, compute $\text{dist}'_A(v, A)$, $\text{dist}'_B(v, B)$

Dijkstra: $O(n^{3/2} \log n)$ MSSP: $O(n \log n)$

(Bottleneck) \Rightarrow

③ Replace A, B with weighted cliques (DDG)
Run Bellman-Ford to compute $\text{dist}_S(s, S)$

$$\hookrightarrow O(VE) = O(n^{3/2})$$

④ Add edges from S to A, B with weight $\text{dist}_S(s, v)$

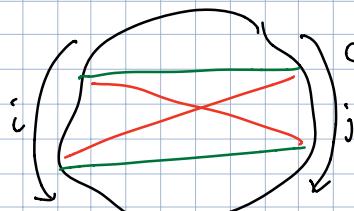
Computed $\text{dist}_S(s, V)$ in $O(n)$ time

$$\begin{aligned} \text{Overall time} \Rightarrow T(n) &= O(n^{3/2} \log n) + T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) \\ &= O(n^{3/2} \log n) \quad \text{MSSP} \end{aligned}$$

[Fakcharoenphol Rao 06]

Monge property!

[Monge 1781]

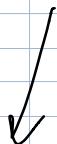


Crossing paths are longer than non-crossing paths

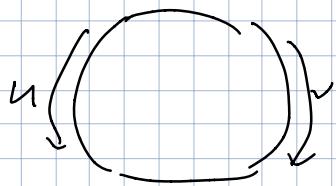
$$M[i, j] + M[i', j'] < M[i, j'] + M[i', j]$$

for all $i < i'$ and $j < j'$

Each iteration of Bellman-Ford



For all vertices v
 For all edges $u \rightarrow v$
 if $\text{dist}(v) > \text{dist}(u) + w(u \rightarrow v)$
 relax($u \rightarrow v$)

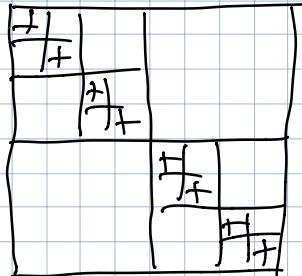
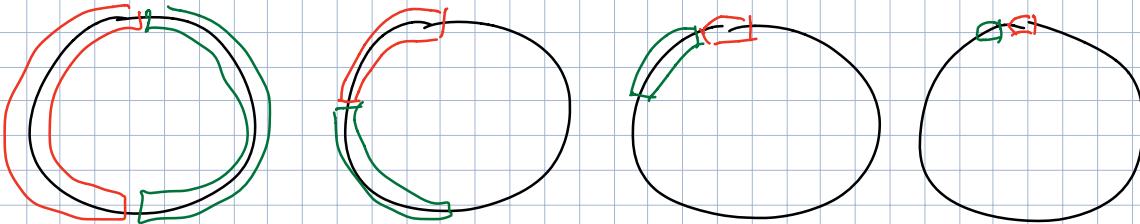


For all v
 $d \leftarrow \min_u (\text{dist}(u) + w(u \rightarrow v))$ ← Let $M[u,v] = \text{dist}(u) + w(u \rightarrow v)$
 if $\text{dist}(v) > d$
 relax

This matrix is Monge
 we need min element
 in each row

SMWK: Computes min dist in every row
 of an $m \times n$ Monge array in $O(m + n)$ time

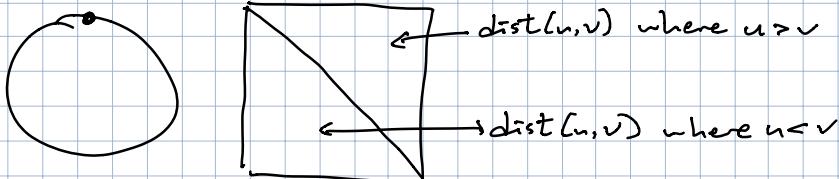
But we don't have a bipartite graph



Submatrices are Monge
 Total time for SMWK is

$$T(n) = O(n) \leftarrow 2T(\frac{n}{2}) = O(n \log n)$$

Even better: We really have a circular dist matrix



Split into two Monge arrays

Klare Kleitman: row min in
 $O(n \alpha(n))$ time

Now Bellman-Ford runs in $O(n)$ iterations
 each $O(n \alpha(n))$ time $\Rightarrow \underline{O(n \alpha(n))}$!