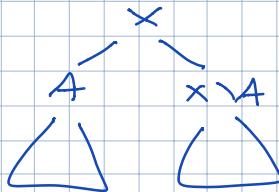


Branchwidth and approximation

A carving of a set X is a maximal family of nonempty noncrossing subsets of X

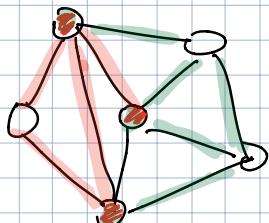
noncrossing means $A \cap B = A$ or B or \emptyset

Every carving contains $2^{|X|}-1$ sets:



Branch decomposition of $G = (V, E)$
= carving of E

For any subset $S \subseteq E$, define $\partial S = \text{vertices incident to both } S \text{ and } E \setminus S$



width of branch decomp. β

$$:= \max_{S \in \beta} |\partial S|$$

branch width of $G =$

min width of branch decomp.

Dynamic programming for NP-hard stuff

Suppose we are given a graph G and a branch decomp β of width β

Ex:

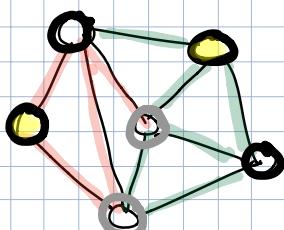
Max Indep Set(G, β):

$O(E \cdot 4^\beta)$ time

Subproblems specified by subset $S \in \beta$ (bit vector describing)

independent set $I \subseteq \partial S$

$$\frac{2^{S-1}}{\leq 2^\beta}$$



Suppose β splits $S = A \cup B$

Subproblem (S, I) depends on $\leq 2^\beta$ subproblems (A, J) and (B, K)

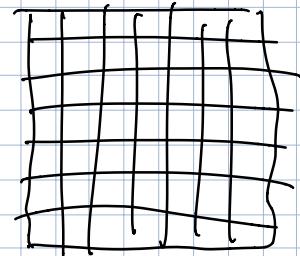
Branchwidth of planar graphs can be computed
in $O(n^2 \log n)$ time

Optimal branch decomp. in $O(n^3)$ time.

But worst-case branchwidth for planar graphs is $\Theta(\sqrt{n})$

And those NP-hard problems are
still NP-hard for planar graphs!

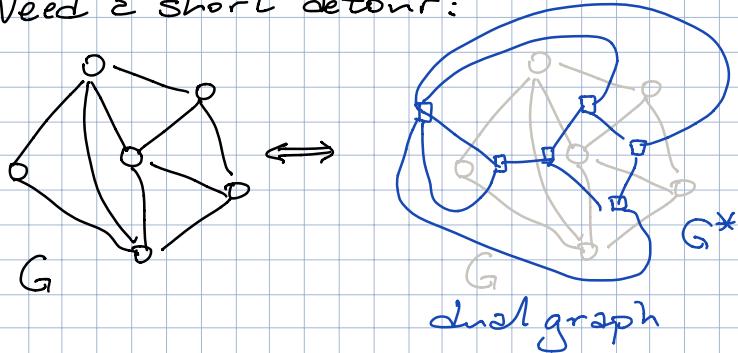
BUT! We can design PTASes for
planar graphs by splitting them
into subgraphs with small
branchwidth



Polyomial-Time Approximation Scheme:

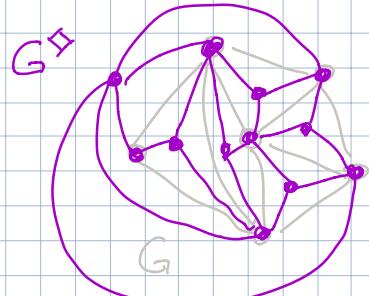
For any ϵ , a poly-time $(1+\epsilon)$ -approx. algorithm.

Need ϵ short detour:

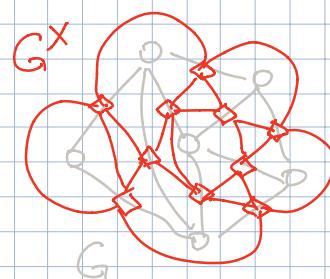


We define two new derived graphs from G :

- Radial graph G^\diamond — $V^\diamond = V \cup F$, $E^\diamond =$ corners
- Medial graph G^X — $V^X = E$, $E^X =$ corners



Every face has degree 4
= quadrangulation



Every vertex has deg. 4
= (multi) curve!

Easy observations: $G^\diamond = (G^*)^\diamond$ $G^x = (G^*)^x$ $(G^\diamond)^* = G^x$
 $\text{dist}^\diamond(u, v) \leq 2 \text{dist}(u, v) \Leftrightarrow \text{diam}(G^\diamond) \leq 2 \min\{\text{diam}(G), \text{diam}(G^*)\}$

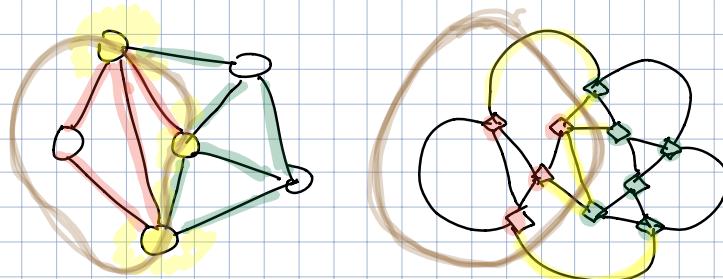
Carving decomposition of G = carving of V

For any subset $S \subseteq V$, $\delta(S) = \text{edges with one endpoint in } S$
("crossing the cut")

Width of carving decompos. $G = \max_{S \in C} |\delta(S)|$
carving width = $\min_C \text{width}(G)$

- Branch decomposition of G = carving decomposition of G^x

Fix a subset $S \subseteq E$



$$\delta(S) = \# \text{corners in } G \text{ incident to } S \text{ and } E \setminus S \geq 2 \cdot \partial(S)$$

\uparrow \uparrow
cut edges in G^x vertices in G

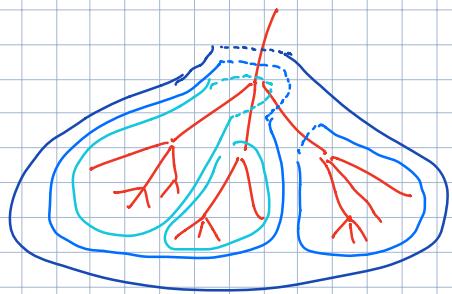
$$\text{So branchwidth}(G) \leq \frac{1}{2} \text{carvingwidth}(G^x)$$

Lemma: $\text{carvingwidth}(G^x) \leq 2 \cdot \text{radius}(G^\diamond)$

Proof: Let $T = \text{BFS tree of } G^\diamond$ [with diameter d]

$C = \text{complementary spanning tree of } G^x$
rooted at arbitrary leaf

Subtrees of C almost define a carving decompos. of G^x



Every node in C has ≤ 3 children

Define up to 3 subsets for each node e

S_e • All descendants of e

S_e^- • All descendants except one subtree

S_e^{--} • All descendants except two subtrees

$$\delta(S_e) \leq d+1 \quad \delta(S_e^-) \leq d+2 \quad \delta(S_e^{--}) \leq d+3$$

$$\Rightarrow \text{carving width}(G^*) \leq d+3$$

$$\Rightarrow \text{branchwidth}(G) \leq \frac{d+3}{2} + 1$$

$$\leq \min\{\text{diam}(G), \text{diam}(G^*)\} + 3$$

Can be removed with more effort

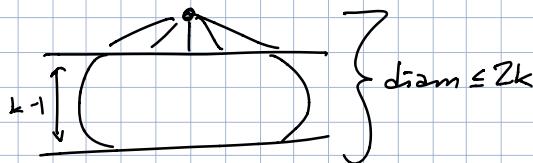
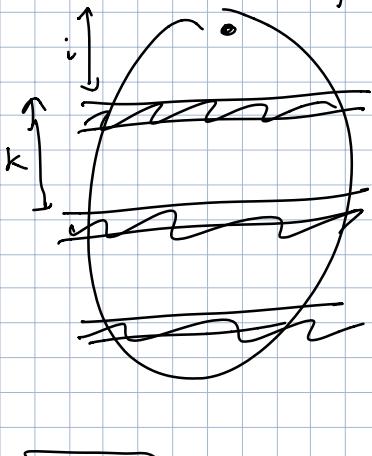
Now back to planar graphs G

Build BFS tree from root r

L_i = verts at distance i

$$L_{ik} = \{v \mid \text{dist}(r, v) \bmod j = k\}$$

• $G[L_{ik}]$ has bw $\leq 2k+1$



Let $\text{OPT} = \text{Max Ind Set}(G)$

$$\text{OPT}_i = \text{OPT} \cap L_{ik}$$

Then $|\text{OPT}_i| \geq \frac{k-1}{k} |\text{OPT}|$ for some i

We can compute MIS($G[L_{ik}]$) in $n \cdot 2^{O(k)}$ time

$$\max_i \text{MIS}(G[L_{ik}]) \geq |\text{OPT}_i| \geq \frac{k-1}{k} |\text{OPT}|$$

Now set $k = \frac{1}{\epsilon}$ We've computed $(1-\epsilon)$ -approx in $n \cdot 2^{O(1/\epsilon)}$ time