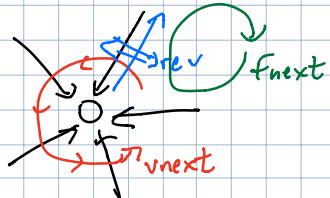


Surface maps / combinatorial surfaces

Recall rotation systems (self-dual incidence lists) for planar embeddings:

Three permutations: $v_{\text{next}}, \text{rev}, f_{\text{next}} : D \hookrightarrow D$



rev = involution w/o fixed points

$$f_{\text{next}} = \text{rev} \circ v_{\text{next}}$$

vertices = orbits of v_{next} (ccw around head)

edges = orbits of rev

faces = orbits of f_{next} (cw around right)

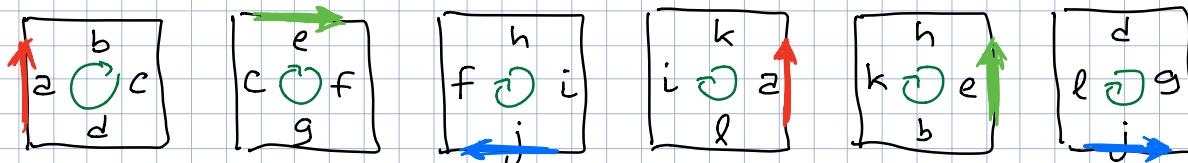
$$\text{Planar} \Leftrightarrow V - E + F = 2 \quad (\text{and connected})$$

But what if $V - E + F \neq 2$?

Then the rot system still describes an embedding, just not on the sphere/plane.

Polygonal schema:

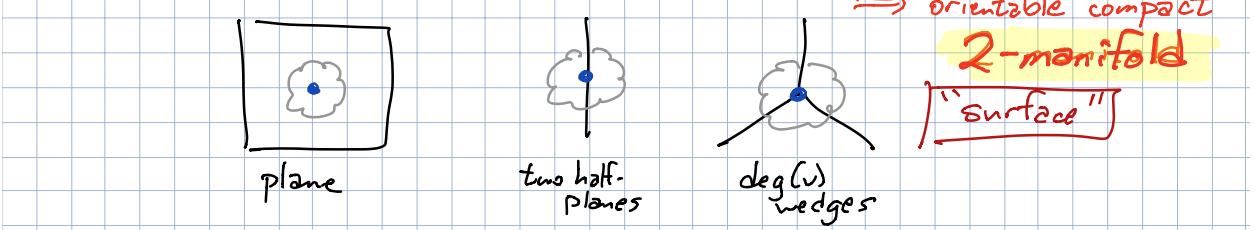
For each Face (orbit of f_{next}), label edges of a "polygon" with darts

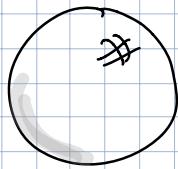


Each edge appears on boundary twice

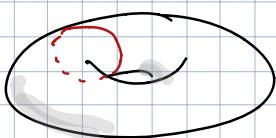
Now glue disks together at corresponding darts
(identify $\text{rev}(d)$ with reversal of d')

The resulting space locally resembles the plane everywhere

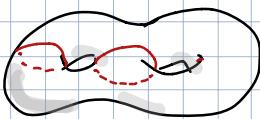




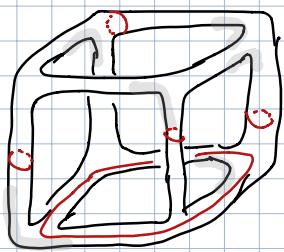
Sphere



torus



There are infinitely many (compact orientable) 2-manifolds, distinguished by genus
 (intuitively, # "handles", but that's not really a thing)



← This surface has genus 5

Formally, genus = max # disjoint simple closed curves whose complement is connected

Tree-cotree decompositions

Recall that when G is planar,

if T = spanning tree of G , then $(E \setminus T)^*$ is spanning tree of G^*

No longer true for surfaces except sphere

- We can contract any edge with distinct endpoints (else it's a loop)
- We can delete any edge separating distinct faces. (else it's an isthmus)

Contract non-loops until only one vertex
 \Rightarrow spanning tree T

Delete non-isthmuses until only one face
 \Rightarrow dual spanning tree = spanning cotree C

But now there might be leftover edges L
 These are all loops and isthmuses

Easy induction argument \Rightarrow

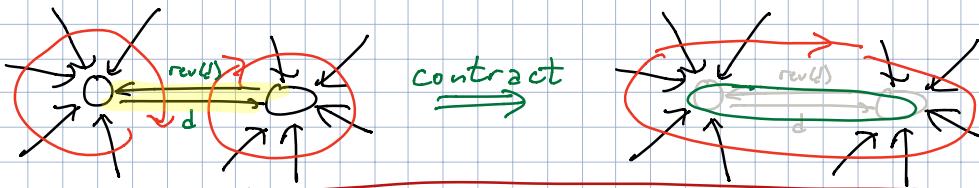
$$V - E + F \leq 2$$

$\hookrightarrow = 2 - L$

Remaining edges define a system of loops

= map with one vertex (base point) and one face

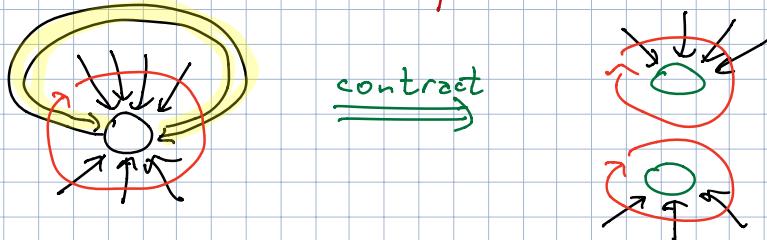
[What actually happens if we delete an isthmus/contract a loop?]



$vnext(vprev(d)) \leftarrow vnext(rev(d))$
 $vprev(vnext(d)) \leftarrow vprev(rev(d))$

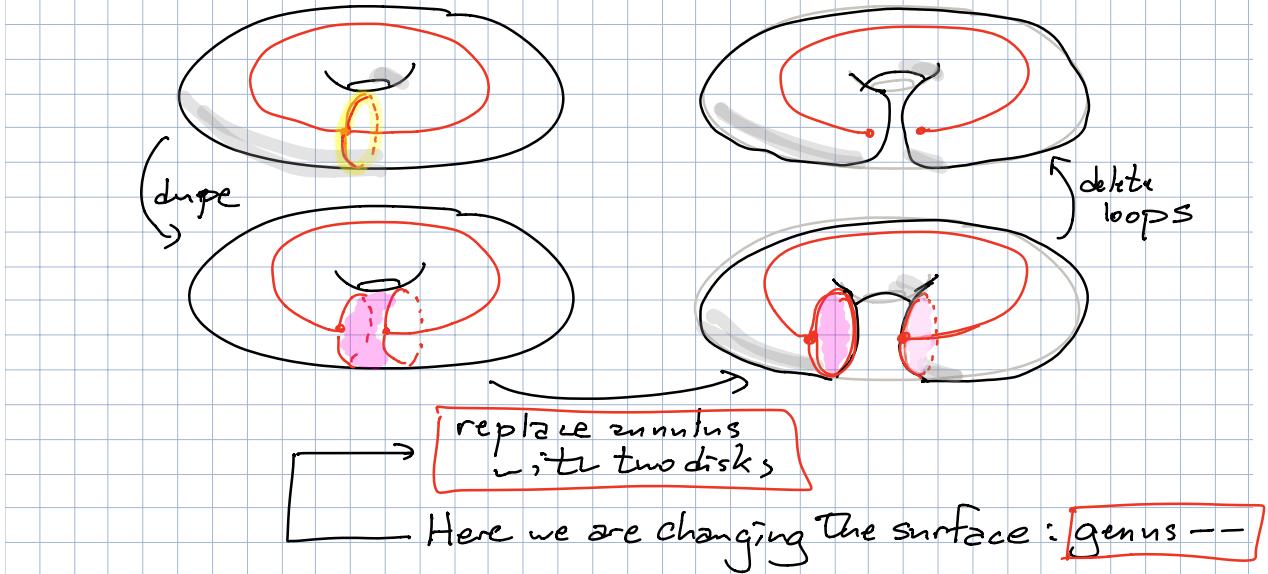
$vnext(vprev(rev(d))) \leftarrow vnext(d)$
 $vprev(vnext(rev(d))) \leftarrow vprev(d)$

But contracting a loop splits the vertex!



(Similarly, deleting an isthmus splits the face!)

On the surface, we're "cutting a handle":

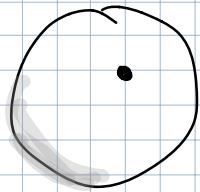


We decrease L by 1, but increase $X = V - E + F$ by 2.

$$\begin{array}{c} +1 \\ \uparrow \\ +1 \end{array} \quad \begin{array}{c} -1 \\ \uparrow \\ -1 \end{array}$$

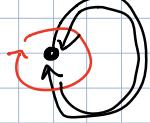
Contracting a non-loop gives us a new system of $L-2$ loops

Base cases: $L=0$



trivial map
of the sphere

$$\cancel{L=1}$$



only rot system has
two darts $d, \text{rev}(d)$
 $v_{\text{next}} = \text{rev}$
 $\Leftrightarrow f_{\text{next}} = \text{id}$
 $\Leftrightarrow \underline{2 \text{ faces!}}$

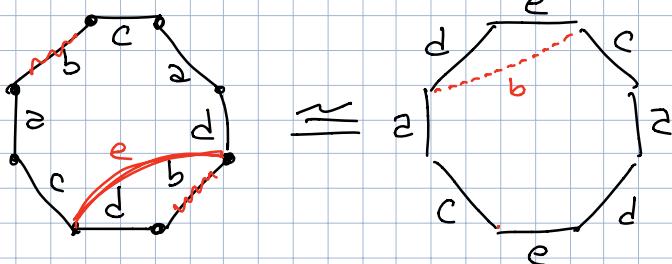
Conclusion: $\boxed{V - E + F = 2 - 2g}$ (in particular always even)

Kerekjártó/Tzadó Theorem:

Every compact 2-manifold is the space of some combinatorial map.

\Rightarrow Every compact orientable 2-mfld is the space of some systems of loops

Brahana: Two systems of loops with the same loops describe homeomorphic surfaces



Canonical poly schema: $\overbrace{abab} \overbrace{cdcd} \dots \overbrace{wxwx} \dots$
 $\textcircled{ab} \overbrace{cd} \dots \overbrace{wx} \overbrace{ab} \overbrace{cd} \dots \overbrace{wx}$

$O(n \log n)$ time [Vegter]

Two generalizations

① Non-orientable surfaces.

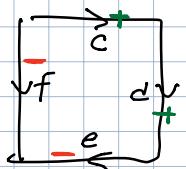
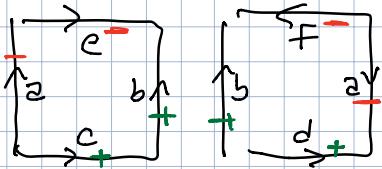
Suppose we allow "turning over" the paper



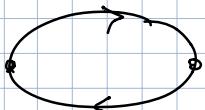
Gauss
Möbius / Listing / al-Jazari
[1858] [1847] [1206] Slade
[1870s]



Klein bottle [1882]



Fortunatus' Purse
[Lewis Carroll 1893]



Projective plane

Represent by a signed polygonal schema / signed rot system

Add fsign: $D \rightarrow \{-1, +1\}$ indicating gluing rule:



Or vsign: $D \rightarrow \{-1, +1\}$ indicating whether "counterclockwise" ordering at end pts is consistent or not

Relationship between vsign and fsign is a bit ugly.

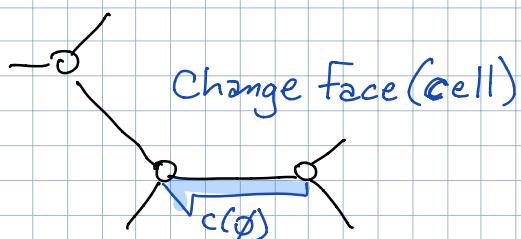
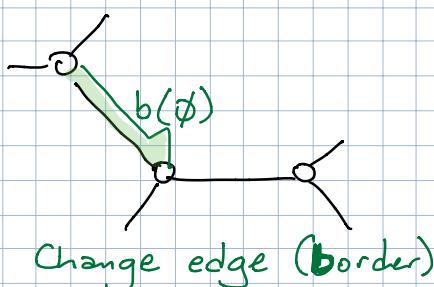
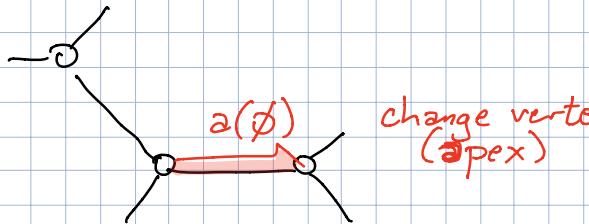
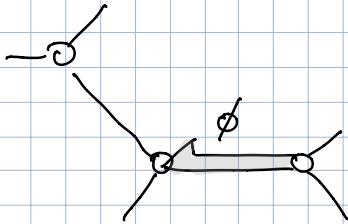
No more correspondence between direction (head/tail) and orientation (left/right)

More natural to regard each edge as four objects: flags or blades

Flag = (vertex, edge, face) incidence

Reflection system (Φ, α, b, c)

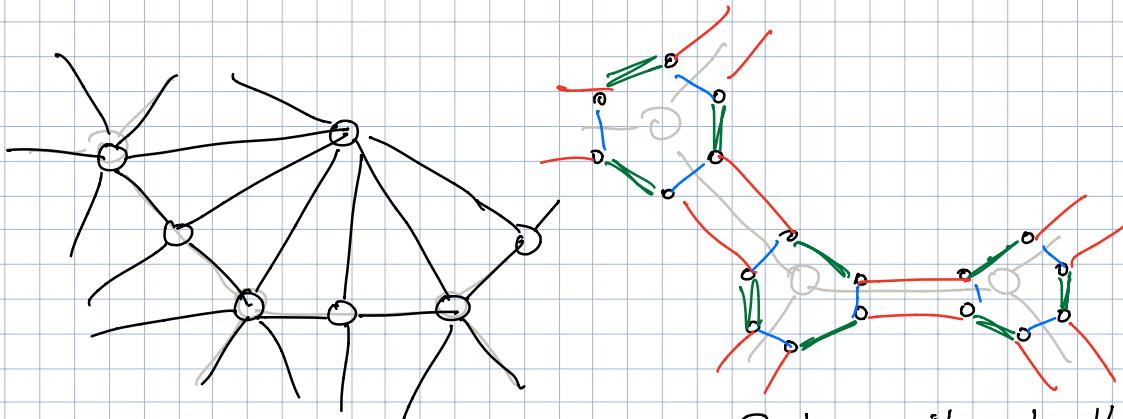
$\Phi = \text{flags}$ $\alpha, b, c : \Phi \leftrightarrow \Phi$ involutions
s.t. $\alpha \circ b = c \circ \alpha$



Change edge (border)

vertices = orbits of $\langle b, c \rangle$
edges = orbits of $\langle \alpha, c \rangle$
faces = orbits of $\langle \alpha, b \rangle$

Blades = faces of barycentric subdivision G^+
= vertices of band decomposition G^\square



Classification: Every compact surface = {sphere with g handles
sphere with h twists}

Euler's formula: $V - E + F = 2 - 2g - h$

$$(g, h) = (g-1, h+2)$$

if $h > 0$



② Surfaces with boundary/punctures