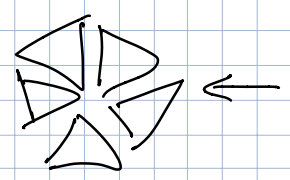
Surfaces ← combinatorial

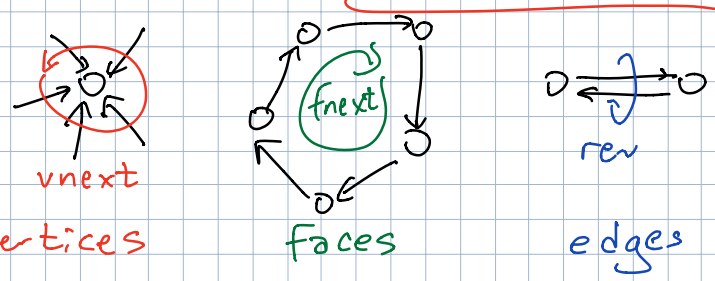
A compact Hausdorff space locally homeomorphic to $\mathbb{R}^2$



Rotation system / self-dual incidence list

Objects — darts

3 permutations $\boxed{\text{vnext, rev, fnext}}$ fnext = rev∘vnext



Planar ⟺ connected, $\boxed{V-E+F=2}$

Combinatorial map surface = rotation system but not nec. planar

= embedding on some surface

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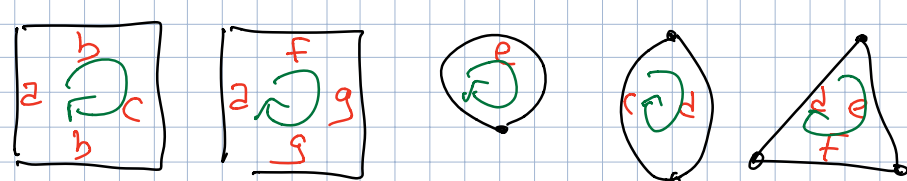
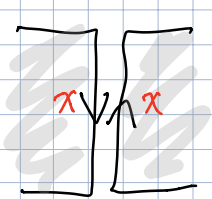
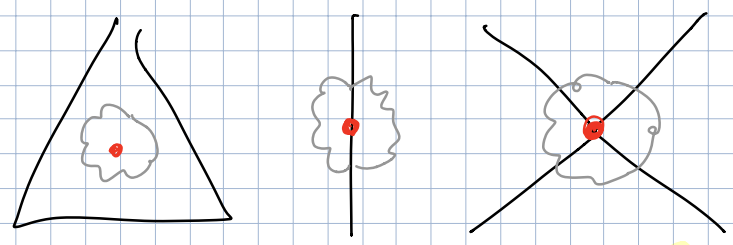
use fnext → polygonal schema



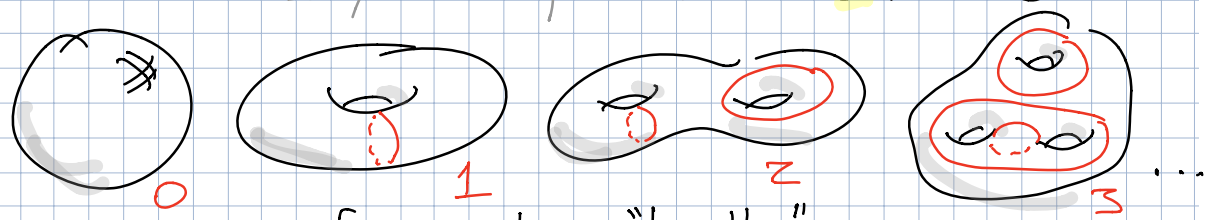
Glue corresponding edges



without turning over the paper



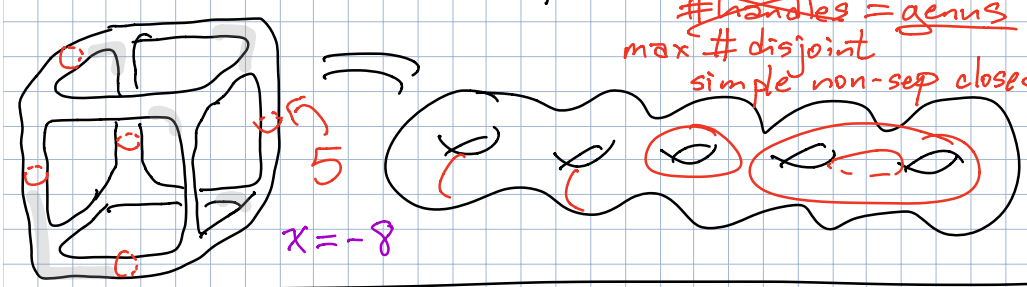
locally homemorphic to $\mathbb{R}^2$ — 2-manifold



surface = sphere + "handles"

~~# handles~~ = genus

max # disjoint simple non-sep closed curves



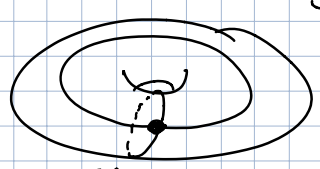
Tree-cotree decomposition

Planar: $E = \underset{\text{sp. tree of } G}{T} \sqcup \underset{\text{sp. tree of } G^*}{C}$

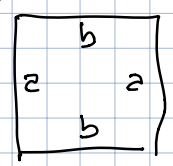
Contract any edge not a loop (two endpts) — T

Delete any edge not an isthmus (separates two faces) — C

Leftover edges — loops <u>and</u> isthmuses



$\chi = 0$

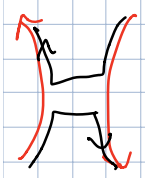


Contr. + Del preserve

$V - E + F = 2 - L$

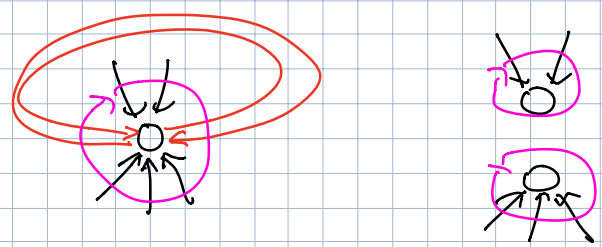
$$\boxed{V - E + F \le 2}$$

Contraction:

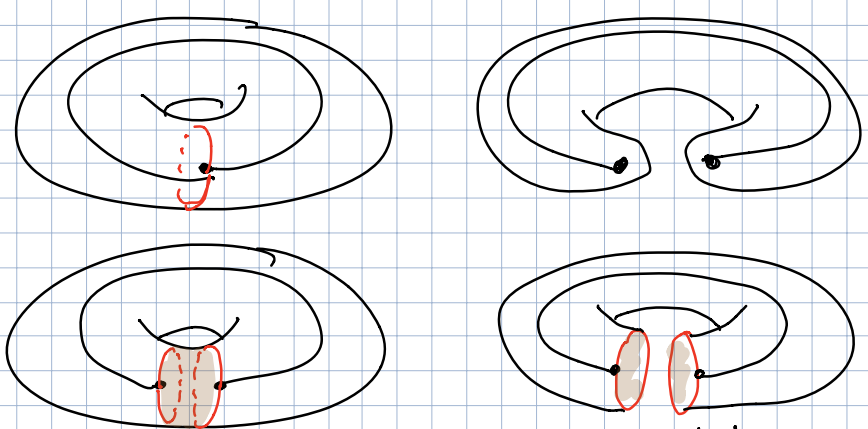


vnext(vprev(d)) ← vnext(rev(d))
vprev(vnext(d)) ← vprev(rev(d))
vnext(vprev(rev(d))) ← vnext(d)
vprev(vnext(rev(d))) ← vprev(d)

If endpts differ → merged!



If one endpoint → split

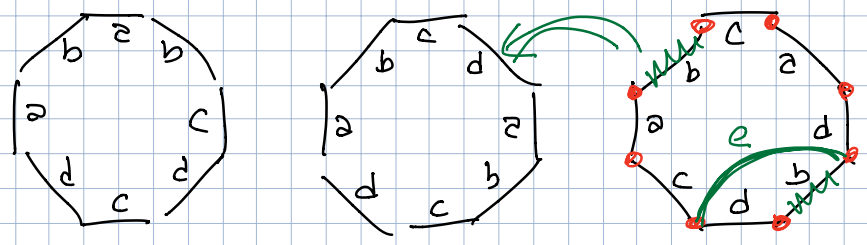


annulus ring → 2 disks

Euler's formula: $V - E + F = 2 - 2g$ ← χ is <u>even</u>

Surface classification: That's all.

[Kerékjártó/Radó 1930s] Every 2-mfld is the space of some comb. embedding



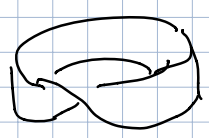
[Brahana '20] [Dehn Heegaard '07] $O(n^2)$

[Vegter '90] $O(n \log n)$ $\longrightarrow$ canonical system of loops for genus n.

<u>Two generalizations:</u>

① Non-orientable surfaces — weirder

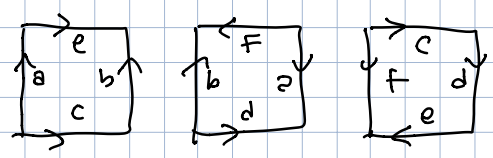
② Boundary/punctures — "throw out faces/vertices"



Möbius band [Möbius 1858 / Listing 1847 / Gauss 1830s / Al-Jazari 1206]

(Klein)

Purse of Fortunatus [Lewis Carroll 1895]



Represent using <u>signed</u> rot sys $\quad$ vsign: $D \to \pm 1$

signed poly schema $\quad$ fsign: $D \to \pm 1$