

# Homotopy on surfaces Dehn [1912]

with boundary — easy

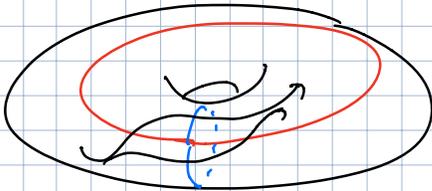
without boundary  
harder — no fences!

build a system of arcs

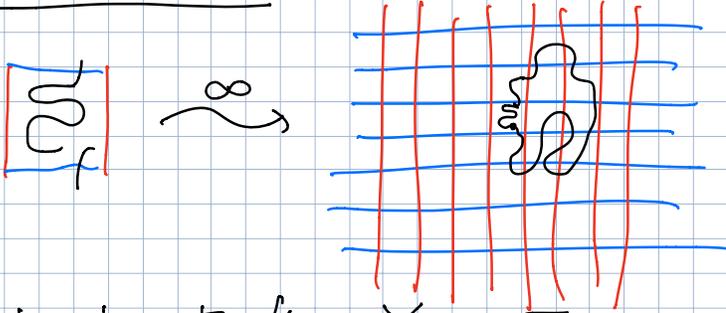
curve  $\rightarrow$  crossing seq

reduce just like polygons with holes

$O(gn + gl)$   
 $\uparrow$  preprocess  
 $\uparrow$  cross seq



Universal cover



closed path/loop  $\gamma$  in  $\Sigma$  is contractible  
iff corresponding path  $\tilde{\gamma}$  in  $\tilde{\Sigma}$  is closed

Strategy: Find a description of  $\tilde{\Sigma}$   
and check if it's closed

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Input: combinatorial map  $\Sigma = (V, E, F)$  — complexity  $n$

closed walk  $\gamma$  in  $(V, E)$  — length  $l$

Output: Is  $\gamma$  contractible in  $\Sigma$ ?

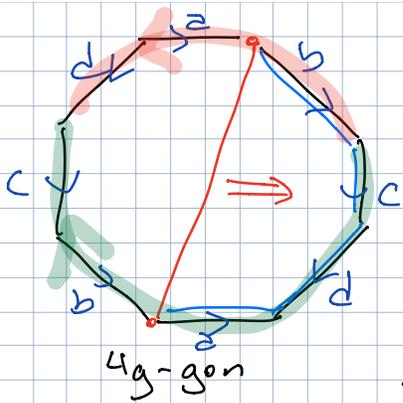
Reduce  $\Sigma$  to a system of loops

$(T, L, C)$  — tree-cotree decomp.

• contract every edge in  $T$  — also contract in  $\gamma$

• delete every edge in  $C$  — reroute  $\gamma$  around face

• left with  $2g$  edges  $L$  — loops + isthmuses



Result:  
 system of loops - complexity  $O(g)$   
 closed walk with length  $O(g)$

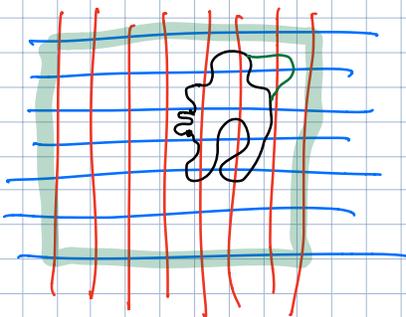
encoding:  $abc\bar{d}dabc\bar{c}\dots$

Two equivalences:

- spurs:  $a\bar{a} = \bar{a}a = \epsilon$
- face:  $abcd\bar{a}\bar{b}\bar{c}\bar{d} = \epsilon$   
 $cd\bar{a}\bar{b}\bar{c} = \bar{b}\bar{a}\bar{d}$

Dehn's lemma - some equivalence shortens the string  
 (if it's contractible)

$g=1$

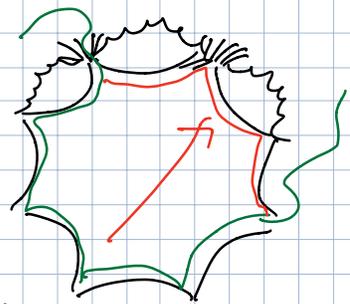
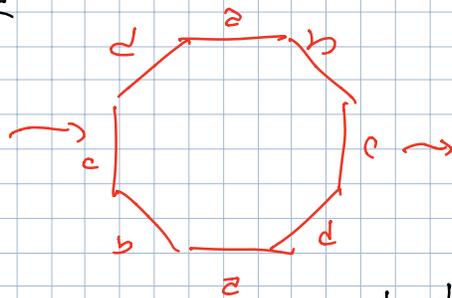
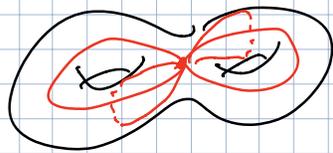


$bbba\bar{b}a\bar{b}\bar{b}\bar{b}\bar{a}b\bar{a}b\bar{a}\bar{b}\bar{b}a$

$a\bar{b}\bar{a}\bar{b} = \epsilon$      $a\bar{b} = \bar{b}a$      $\bar{a}\bar{b} = \bar{b}\bar{a}$

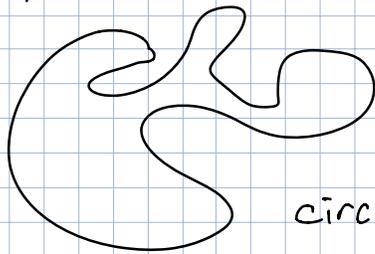
contractible  $\Leftrightarrow \#a=0$      $\#b=0$

$g>1$  Universal cover



regular tiling of the hyperbolic plane

# Isoperimetric inequality



length  $l \Rightarrow$

Euclidean: Area  $\leq O(l^2)$

hyperbolic: Area  $\leq O(l)$

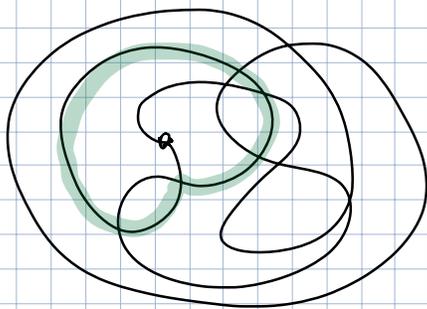
circle of radius  $r$

Euclidean: perimeter  $2\pi r$   
area  $\pi r^2$

Hyperbolic: per.  $c^r$   
area  $\Theta(c^r)$

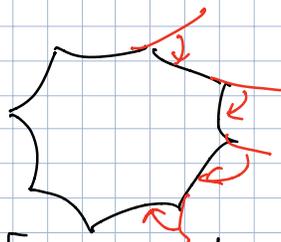
Dehn's lemma: Any <sup>nonempty</sup> closed walk in hyperbolic tiling contains a spur or strict majority of a tile boundary

Proof: WLOG assume walk is simple



Consider a disk in tiling  $D$

Assign an exterior angle  $\angle_c$  to each corner of  $D$

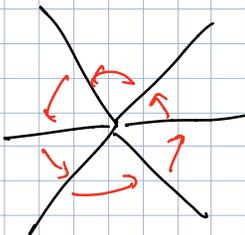


Define curvature of  $f$

$$\kappa(f) = 1 - \sum_{c \in f} \angle_c$$

Define curvature of  $v$

$$\kappa(v) = 1 - \frac{1}{2} \deg(v) + \sum_{c \in v} \angle_c$$



**Discrete Gauss-Bonnet**

$$\sum \kappa(f) + \sum \kappa(v) = \chi = 1$$

$$F - \sum \angle_c \quad V - E + \sum \angle_c$$

set  $\angle c = \frac{1}{4}$  at every corner

$$\kappa(F) = 1 - 4g\left(\frac{1}{4}\right) = 1 - g < 0$$

$$\kappa(v) = \begin{cases} \leq 0 & \text{if } v \text{ interior} \\ \frac{2}{4} & \text{if } v \text{ boundary, one int corner} \\ \leq 0 & \text{other boundary} \end{cases}$$

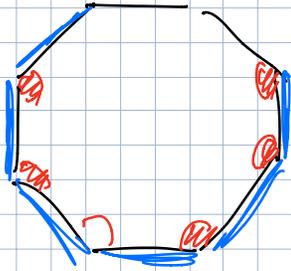
$$\Rightarrow F(1-g) + V_+ / 4 \geq 1$$

$$V_+ \geq 4(g-1)F + 1$$

some face has  $\geq 4g-3$  convex vertices

↑  
must be consecutive

$\Rightarrow 4g-2$  consecutive boundary edges



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Algorithm: Scan thru edge seq, reduce when possible

Naively - check  $O(g)$  strings of length  ~~$g+1$~~   $4g-2$  at every char.

$\Rightarrow O(g^3 l)$  time

DFA -  $O(g^2)$  prep

$O(1)$  time per char.

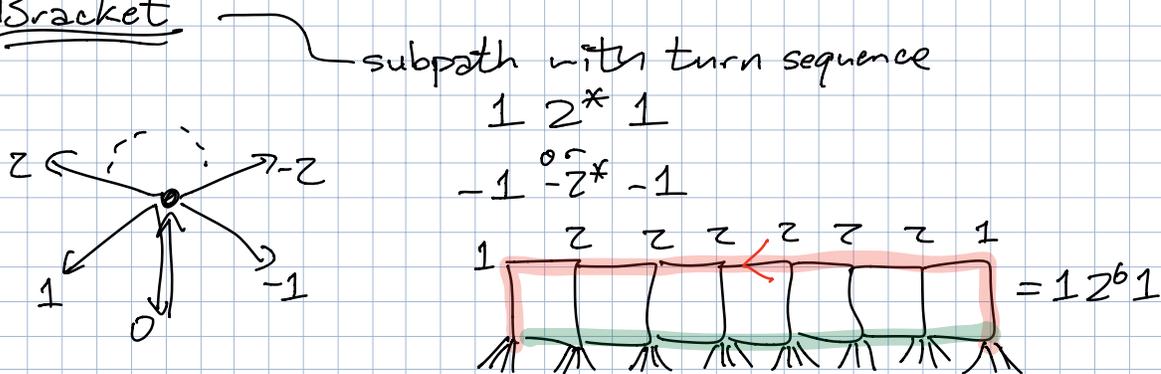
$\Rightarrow O(gl)$  time

$O(n+1)$  time [RivardLazarus 13 / Ewhittleay 14]

System of loop  $\rightarrow$  face is too big

Instead reduce to radial graph of sys. of loops  
system of quads

Bracket



① Every closed walk in quad tiling contains a spur or bracket

② Run-length encoding  $\rightarrow$  spend  $O(1)$  time per edge of  $\delta$