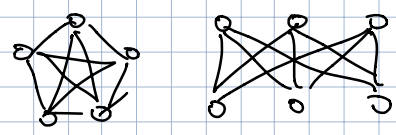
Graph Minors

subgraph — result of deleting edges + vertices

minor — result of del/contracting edges

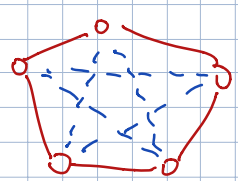
Kurotowski/Wagner

A graph is planar $\Leftrightarrow$ no K_5 or $K_{3,3}$ minor



Lemma: Any minor of planar graph is planar

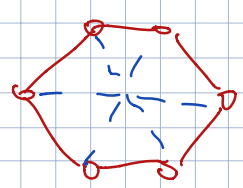
Lemma: K_5 and $K_{3,3}$ aren't planar



interlacement graph
verts = chords

edges =

$IG = K_5$ not bipartite



$IG = K_3$ not bipartite

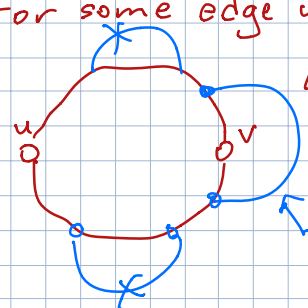
$\square$

Lemma: Nonplanar $\Rightarrow$ K_5 or $K_{3,3}$ minor

Let G minimal nonplanar
verts have degree ≥ 3

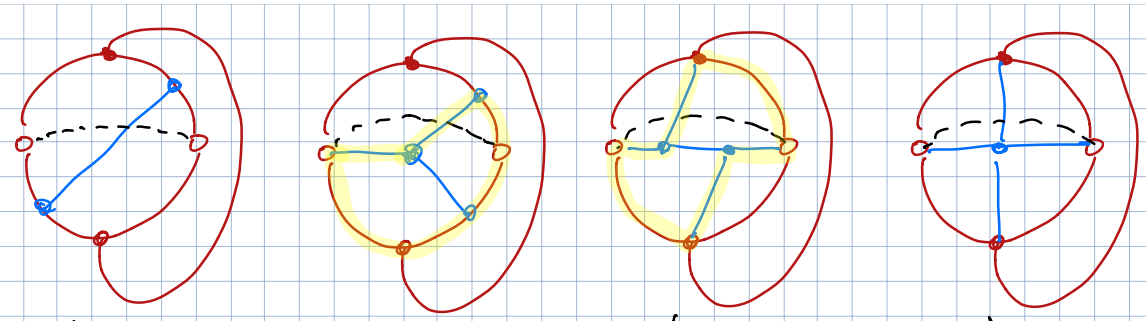
2-connected

For some edge uv, subgraph $G \setminus uv$ is 2-connected



Let C be max area (#faces) cycle
over <u>all</u> planar emb of $G \setminus uv$

Must exist



$K_{3,3}$ $K_{3,3}$ $K_{3,3}$ K_5

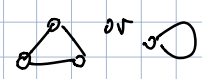
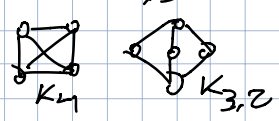
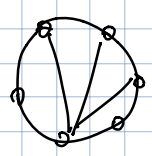
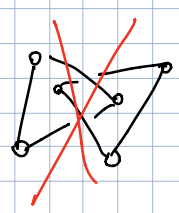
Minor-closed $\Longleftrightarrow$ Finite set of forbidden minors

Robertson
Seymour

Alg. to decide if H is minor of G in $O(n^3)$ time

$\#V(H)$ $\#V(G)$

$\Rightarrow$ Alg to decide if G is in fixed minor-closed family in $O(n^3)$ time

Family	F. Minors
forests	
planar	K_5 $K_{3,3}$
outerplanar	 K_4 $K_{3,2}$
	
 linklessly embeddable	7
projective plane	35
torus	≥ 16629 — 68 2-connected; 4 $K_{3,3}$-free
treewidth k	every forbidden minor has $\leq 2^{2^{O(k^4)}}$ vertices.

$2\Uparrow(2\Uparrow(2\Uparrow(n/2)))$

$n = \#V(H)$

Ugh.

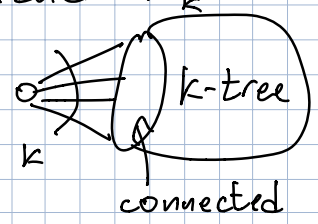
$2\Uparrow 0 = 1$

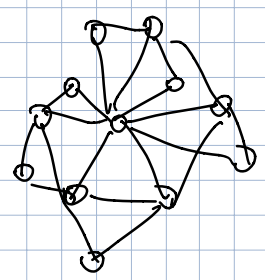
$2\Uparrow k = 2^{2\Uparrow(k-1)}$

Treewidth

k-tree is either K_k

or





Graph has treewidth k

$\iff$ subtree of k-tree not (k-1) tree

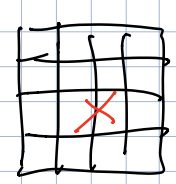
Fast DP algorithms

↓

PTAS for surface graphs

Planar $\Rightarrow$ tw $O(\sqrt{n})$

$n \times n$ grid $\Rightarrow$ tw $= n$



RS: There is a function $f: \mathbb{N} \to \mathbb{N}$:

If G has no $r \times r$ gridminor, then

$tw(G) \le f(r)$

$f(r) = \Omega(r^2 \log r)$ $\quad$ ~~$2^{O(r^5)}$~~ $\quad$ $O(r^{36} \text{ polylog } r)$

Chekuri Chuzhoy 16

Graph Structure Theorem

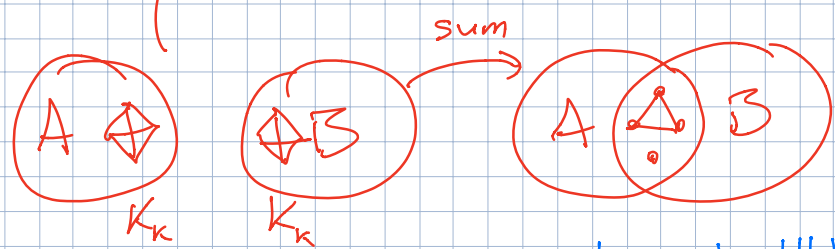
For any graph H there is an integer $k = k(H)$

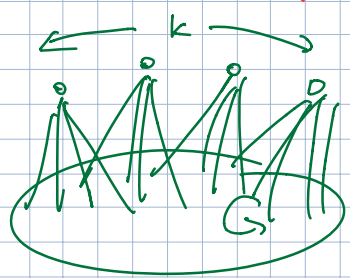
any H-minor free graph is

a $\boxed{k\text{-clique sum}}$ of finite # of

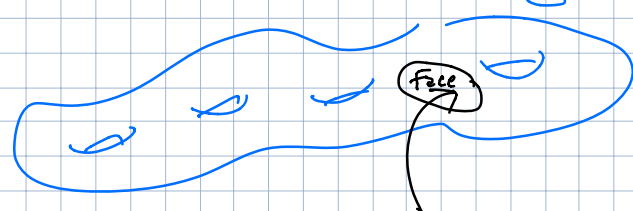
$\boxed{k\text{-apex graphs}}$ of

$\boxed{k\text{-almost embeddable}}$ graphs





k-embeddable = emb. on genus k



Vortex: graph with pathwidth k