

Trivial Closed Walks in Directed Surface Graphs

Motivation: Lots of algorithms start with greedy tree-co-tree

Fails if graph has negative cycles

But some problems still solvable w/ negative cycles:

- Shortest homotopic path
- Min-wt homologous subgraph/circulation

Input: directed graph $G=(V,E)$ n vertices $g=O(n)$
embedded on surface Σ genus g , b boundaries, orientable

$$\text{First Betti \# } \beta = \begin{cases} 2g & \text{if } b=0 \\ 2g+b-1 & \text{if } b>0 \end{cases}$$

Is any closed walk in G contractible?

If so, find the shortest such walk. $\leftarrow \mathbb{Z}$ coefficients (\mathbb{Z}_2 is trivial)

[Why "closed walk"? Because "cycle" is NP-hard!]

First problem is trivial if G has anti-homotopic edges \rightleftarrows
or contractible loops \odot
so assume otherwise.

Results:

- Detect ccw in $O(n)$ time
bcw $O(gn)$ time
- Shortest ccw in $O(\beta^5 n^3)$ time when $b>0$
 $O(\beta^3 n^7 \log^2 n)$ time when $b=0, g \geq 2$ \rightarrow "CFG Dijkstra"
 $O(n^5)$ time when $b=0, g=1$ $O(N \cdot P \cdot n^3)$
- Shortest bcw in $O(n^{2\beta+2})$ time
- Negative bcw in $O(\beta^5 n \log^2 n \log^2 L)$ time [integer wts $-U \leq e \leq U$]
- Negative ccw in $2^{\text{poly}(n)}$ time when $b>0$
no algorithm known when $b=0, g \geq 2$

Detecting ccws

① Weakly simple

Perturb ccw W into generic curve

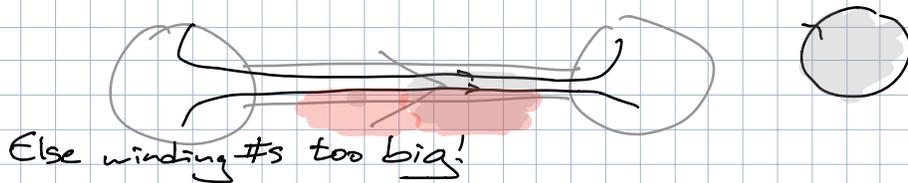
Hass + Scott: Either simple or ^{contr.} monogen or ^{contr.} bigon

- contr. monogen:  \rightarrow shorter ccw

- incoherent bigon:  \rightarrow shorter ccw

- coherent bigon:  \Rightarrow 
 smooth still contractible

② Weakly simple ccw \Rightarrow each edge at most once



③ bcws avoid cocycles $F_1 \uparrow F_2 \uparrow F_3 \uparrow \dots \uparrow F_n \uparrow F_1$

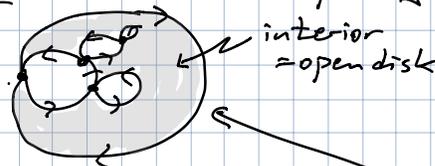
Any bcw \rightarrow boundary circulation \rightarrow Alexander #

Must have $\alpha(F_i) \geq \alpha(F_{i-1})$ for all i

Only possible if $\alpha(F_i) = \alpha(F_{i-1})$ for all i

④ Cocycle-free + ccw \Rightarrow simple face boundary

[but not nec. simple:]



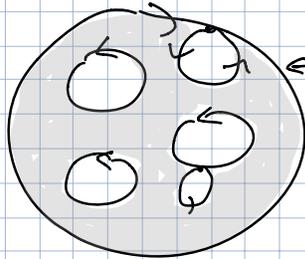
Let W be weakly simple ccw

$A =$ area "inside" W — must be open disk (like this)
 wlog directed clockwise

Dual walk starting in A stays in A
 no cocycle \rightarrow must get stuck at some face F .

If F is a disk, we're done.

o/w F is a disk with holes (because $F \subseteq A$)



\leftarrow holes

Let w' be bdry of any hole

$w' \subseteq A \rightarrow$ contractible

w' encloses fewer faces than w

Lemma follows by induction on # enclosed faces. \square

Algorithm:

Remove all cocycles

For each face F

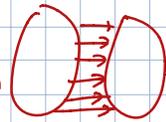
if F is a disk and ∂F is coherent
 return TRUE

return FALSE

$O(n)$ time

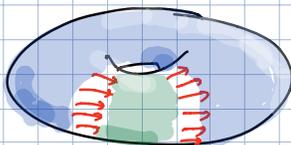
Detecting cuts:

- $w \log G$ is strongly connected \rightarrow no cuts
 (= sum of cocycles)
- $w \log G$ has no cocycles



But separating components can add cocycles to G

deleting cocycles can disconnect G even if G is s.c.



Define $G_0 =$ any strong comp. of G
 $H_0 = G_0 \setminus \text{cocycles}$
 $G_1 =$ any (strong) comp of H_0
 $H_1 = G_1 \setminus \text{cocycles}$
 \vdots

Eventually converges: $G_i = G_{i-1}$ $\xrightarrow{\text{with } \geq 2 \text{ faces} \Leftrightarrow \geq 1 \text{ edge}}$

Lemma: Let G be a dir. surface graph. If G is connected + cocycle free there is a bcw in G

Proof: G^* is a dag. Number faces $\alpha(f)$ by top. order of G^*

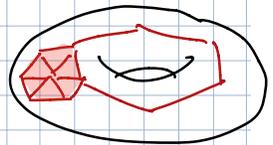
$$\partial \alpha(e) = \alpha(\text{left}(e)) - \alpha(\text{right}(e)) > 0 \text{ for all } e$$

\uparrow positive boundary circulation

Any Euler walk of $\partial \alpha$ is a bcw. \square

- Any cocycle in G_i visits a non-simple face of G_i
- Any simple face of G_i is a simple face of G_0

$[G_i] =$ footprint of $G_i = G_i \cup$ all simple faces of G_i



Lemma: $\beta([G_i]) \leq 2g - i$

Proof: Let λ be a cocycle in G_{i-1} .

\Rightarrow dir. closed curve λ^* crossing edges of λ left to right

λ^* intersects non-simple face of G_{i-1}

so $\lambda^* \cap [G_{i-1}] =$ one or more disjoint bdy to bdy paths = arcs.

No individual arc is separating in $[G_{i-1}]$ because G_{i-1} is str. con.

For any surface Σ with bdy, any arc α

$$\beta_2(\Sigma \setminus \alpha) = \begin{cases} \beta_2(\Sigma) - 1 & \text{if } \alpha \text{ is non-separating} \\ \beta_2(\Sigma) & \text{if } \alpha \text{ is separating} \end{cases}$$

So in fact $\sum_j \beta[G_{i,j}] \leq \beta[G_{c-1}] - 1$

where $G_{i,j}$ are components of H_{c-1} . \square

Theorem: we can detect bcws in $O(qn)$ time.