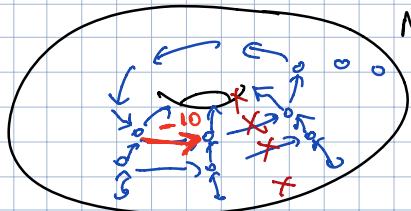
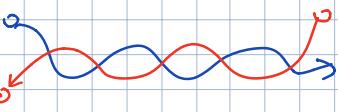
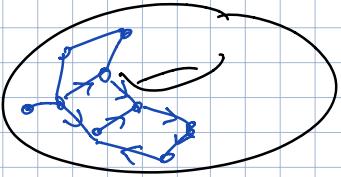


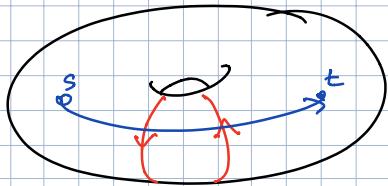
Finding Trivial Closed Walks in Directed Surface Graphs



No nontrivial closed walks

Shortest homotopic paths well-defined unless

- contractible neg cycle
- anti-homotopic neg cycle pair



Given dir. graph $G = (V, E)$

embedded on some orientable surface genus $g = O(n)$

Is any closed walk in G contractible?
null-homologous?

CCW $O(n)$
BCW $O(gn)$

"cycle" \Rightarrow NP-hard [Cabello]

No contractible loops

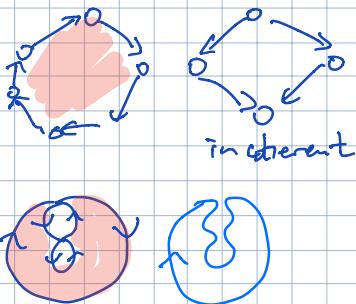
anti-homotopic edges

CCWs in $O(n)$ time

- Remove any cocycles from G
- For every face F if F is a disk and ∂F is coherent return True
- return FALSE

$F_1 \uparrow F_2 \uparrow F_3 \uparrow \dots \uparrow F_k \uparrow F_1$

$v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1$

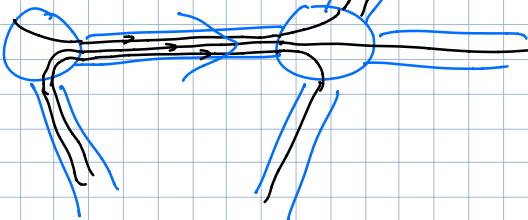


Lemma: If G has a ccw, then G has a weakly simple ccw
 arbitrarily close to a simple cycle

Proof: Let ω be any ccw.

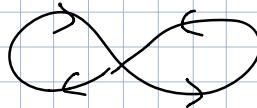
ω be any generic perturbation of ω

If ω is simple, done.



[Hass Scott]:

There must be a monogen or a bigen



□

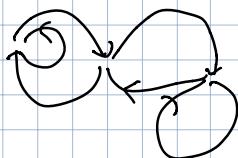
Lemma: weakly simple ccw uses each edge at most once

Proof: [Epstein 66] ω is boundary of a disk

Label faces of G in or out



□

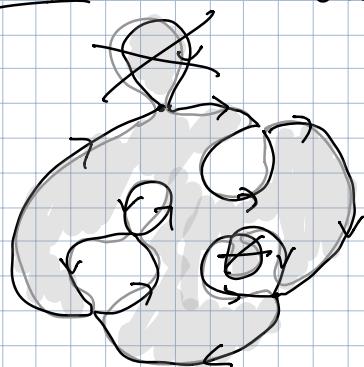


$\Rightarrow \text{length} \leq O(n)$

Lemma: If G has a weakly simple ccw, then G has a simple face with coherent bdry

Proof:

WLOG ω has connected interior A
 \cong open disk

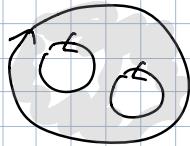


Follow a dual walk $F \uparrow F' \uparrow \dots$
 inside A

Get stuck at face F with cw boundary

IF F is a disk \rightarrow done

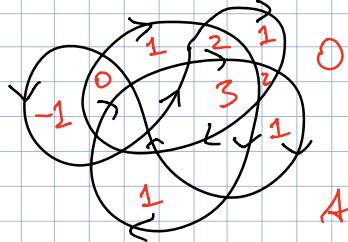
O/w $F \not\subseteq A \Rightarrow F$ is a disk with holes



W' = bdry of any hole

W' is contractible, encloses fewer faces than W

Lemma: No ~~bew~~ uses any edge in any cocycle



Alexander numbering

$\alpha: F \rightarrow \mathbb{N}$ "Z-chain"

$\partial\alpha: E \rightarrow \mathbb{Z}$ $\partial\alpha(e) = \alpha(\text{right}(e)) - \alpha(\text{left}(e))$
boundary circulation

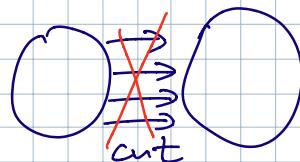
WANT:

- non-negative
- positive edges is connected

Boundary closed walk = Euler tour of
pos. bdry circulation
with connected support

WLOG ① No cocycles — G^* is a dag

② G is strongly connected



$G_0 = \text{any strong comp of } G$

$H_0 = G_0 \setminus \text{cocycles}$

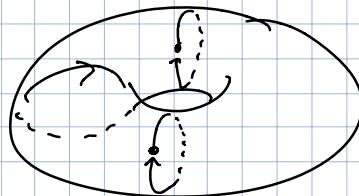
$G_1 = \text{any strong comp of } H_0$

$H_1 = G_1 \setminus \text{cocycles}$

:

:

until $G_i = G_{i-1}$



① IF G_i has more than one vertex
then G_i has bcw

$\Rightarrow O(n)$ time

② Converges after $\leq 2g$ iterations