12 Tutte’s Spring Embedding Theorem

In 1963, William Tutte published a paper ambitiously entitled “How to Draw a Graph”. Let $\Sigma$ be any planar embedding of any simple planar graph $G$.

- Nail the vertices of the outer face of $\Sigma$ to the vertices of an arbitrary strictly convex polygon $P$ in the plane, in cyclic order.
- Build the edges of $G$ out of springs or rubber bands.
- Let go!

Tutte proved that if the input graph $G$ is sufficiently well connected, then this physical system converges to a proper straight-line embedding of $G$ with convex faces!

Let me state the parameters of the theorem more precisely, in slightly more generality than Tutte did. A Tutte drawing of a planar graph $G$ is described by a function $p : V \rightarrow \mathbb{R}^2$ mapping the vertices to points in the plane, subject to the two conditions:

1. The vertices of one face $f_\infty$ of some embedding of $G$ are mapped to the vertices of a strictly convex polygon in cyclic order.
2. Each vertex $v$ that is not in $f_\infty$ maps to a point in the interior of the convex hull of its neighbors; that is, we have
   \[ \sum_{u \rightarrow v} \lambda_{u \rightarrow v} (p_v - p_u) = 0 \]
   for some positive real coefficients $\lambda_{u \rightarrow v}$ on the darts into $v$.

The edges of a Tutte drawing are line segments connecting their endpoints. Let me emphasize that the definition of a Tutte drawing does not require mapping edges to disjoint segments, or even mapping vertices to distinct points.

We call the face $f_\infty$ (or its image under $p$) the outer face of the drawing. We call the vertices of $f_\infty$ boundary vertices, and the remaining vertices of $G$ interior vertices. Similarly, we call the edges of $f_\infty$ boundary edges, and the remaining edges of $G$ interior edges. This terminology is justified by the following observation:

Outer face lemma: In every Tutte drawing, every interior vertex maps to a point in the interior of the outer face. In particular, no interior vertex maps to the same point as a boundary vertex.

Proof: We argue by applying Gaussian elimination to the system of linear equations defined by condition (2). Linear system (2) expresses the position $p_v$ of any vertex $v$ as a strict convex combination of the positions of its neighbors in $G$, that is, a weighted average where every neighbor of $v$ has positive weight. By pivoting on that row, we can remove the variables $p_v$ from the system. This pivot is equivalent to deleting vertex $v$ and adding new edges between the neighbors of $v$, with appropriate positive coefficients on their darts. Thus, if we eliminate all but one interior vertex $w$, the remaining constraint expresses $w$ as a strict convex combination of the boundary vertices.

The same elimination argument implies that every assignment of positive dart coefficients $\lambda_{u \rightarrow v} > 0$ defines a unique Tutte drawing; linear system (2) always has full rank.

---

1The formulation and proof of Tutte’s theorem that I’m presenting here follows a lecture note by Dan Spielman (2018), which is based on papers by Michael Floater (1997); László Lovász (1999); Steven Gortler, Craig Gotsman, and Dylan Thurston (2006); and Jim Geelen (2012).

2This modification is called a star-mesh transformation; the special case of removing a vertex of degree 3 is called a Y-Δ transformation.
A graph $G$ is 3-connected if we can delete any two vertices without disconnecting the graph. Whitney (1932) proved that every simple 3-connected graph $G$ has a unique embedding on the sphere (up to homeomorphism). Thus, in every planar embedding of $G$, the faces are bounded by the same set of cycles; we can reasonably call these cycles the faces of $G$.

**Tutte’s spring-embedding theorem:** Every Tutte drawing of a simple 3-connected planar graph $G$ is a straight-line embedding of $G$ with strictly convex faces.

### 12.1 Laplacian linear systems and energy minimization

Tutte’s original formulation required that every interior vertex lie at the center of mass of its neighbors; this is equivalent to requiring $\lambda_{u\rightarrow v} = 1$ for every dart $u \rightarrow v$. More generally, the physical interpretation in terms of springs corresponds to the special case where dart coefficients are symmetric.

Suppose each edge $uv$ is to a (first-order linear) spring with spring constant $\omega_{uv} = \lambda_{u\rightarrow v} = \lambda_{v\rightarrow u}$. For any vertex placement $p \in (\mathbb{R}^2)^V$, the total potential energy in the network of springs is

$$\Phi(p) := \frac{1}{2} \sum_{u,v} \omega_{uv} \|p_u - p_v\|^2.$$ 

If we fix the positions of the outer vertices, $\Phi$ becomes a strictly convex\(^3\) function of the interior vertex coordinates. If we let the interior vertex positions vary, the network of springs will come to rest at a configuration that minimizes $\Phi$. The unique minimum of $\Phi$ can be computed by setting the gradient of $\Phi$ to the zero vector and solving for the interior coordinates; thus we recover the original linear constraints

$$\sum_v \omega_{uv} (p_u - p_v) = 0$$

for every interior vertex $u$. The underlying matrix of this linear system is called a weighted Laplacian matrix of $G$. This matrix is positive definite\(^4\) and therefore non-singular, so a unique equilibrium configuration always exists.

When the dart coefficients are not symmetric, this physical intuition goes out the window; the linear system of balance equations is no longer the gradient of a convex function. Nevertheless, as we’ve already argued, any choice of positive coefficients $\lambda_{u\rightarrow v}$ corresponds to a unique straight-line drawing of $G$. None of the actual proof of Tutte’s theorem relies on any special properties of the coefficients $\lambda_{u\rightarrow v}$ other than positivity.

Given the graph $G$, the outer convex polygon, and the dart coefficients, we can compute the corresponding vertex positions in $O(n^3)$ time via Gaussian elimination. (There are faster algorithms to solve this linear system. In particular, a numerically approximate solution can be computed in $O(n \log n)$ time in theory, or in $O(n \text{ polylog } n)$ time in practice.)

### 12.2 Slicing with Lines

For the rest of this note, fix a simple 3-connected planar graph $G$ and a Tutte drawing $p$. At the risk of confusing the reader, I will generally not distinguish between features of the abstract graph

\(^3\)The Hessian of $\Phi$ is positive definite, meaning all of its eigenvalues are positive.

\(^4\)The Laplacian matrix is just the Hessian of $\Phi$. 

G (vertices, edges, faces, cycles, paths, and so on) and their images under the Tutte drawing (points, line segments, polygons, polygonal chains, and so on). For example, an edge of the Tutte drawing p is the (possibly degenerate) line segment between the images of the endpoints of an edge of H, and a face of the Tutte drawing p is the (not necessarily simple) polygon whose vertices are the images of the vertices of a face of G in cyclic order.

**Both sides lemma:** For any interior vertex v and any line ℓ through p_v, either all neighbors of v lie on ℓ, or v has neighbors on both sides of ℓ.

**Proof:** Suppose all of v’s neighbors lie in one closed halfplane bounded by ℓ. Then the convex hull of v’s neighbors also lies in that halfspace, which implies that v does not lie in the interior of that convex hull, contradicting the definition of a Tutte drawing. □

**Halfplane lemma:** Let H be any halfplane that contains at least one vertex of G. The subgraph of G induced by all vertices in H is connected.

**Proof:** Without loss of generality, assume that H is the halfplane above the x-axis. Let t be any vertex with maximum y-coordinate; the outer face lemma implies that t is a boundary vertex. I claim that for any vertex u ∈ H, there is a directed path in G from u to t, where the y-coordinates never decrease. There are two cases to consider:

- If t and u have the same y-coordinate, the outer-face lemma implies that either t = u or u is an edge of the outer face. In either case the claim is trivial.

- Otherwise, u must lie below t. Let U be the set of all vertices reachable from u along horizontal edges of G. Because G is connected, some vertex v ∈ U has a neighbor that is not in U. The both-sides lemma implies that v has a neighbor w that has larger y-coordinate than v. The induction hypothesis implies that there is a y-monotone path in G from w ↦ t. Thus, u ↦ v ↦ w ↦ t is a y-monotone path, proving the claim. □

![Figure 1: The halfplane lemma.](image)

### 12.3 No Degenerate Vertex Neighborhoods

None of the previous lemmas actually require the planar graph G to be 3-connected. The main technical challenge in proving Tutte’s theorem is showing that if a G is 3-connected, the faces of any Tutte drawing of G are non-degenerate. The assumption of 3-connectivity is necessary— if G is 2-connected but not 3-connected, then some faces can degenerate to line segments, and if G is connected but not 2-connected, some faces can degenerate to single points.

**Utility lemma:** The complete bipartite graph K_{3,3} is not planar.

**Proof:** K_{3,3} has n = 6 vertices and m = 9 edges, so by Euler’s formula, any planar embedding would have exactly 2 + m − n = 5 faces. On the other hand, because K_{3,3} is simple and
bipartite, every face in any planar embedding would have degree at least 4. Thus, a planar embedding of \( K_{3,3} \) would imply \( 20 = 4f \leq 2m = 18 \), which is obviously impossible. \( \square \)

**Nondegeneracy lemma:** No vertex of \( G \) is collinear with all of its neighbors.

**Proof:** By definition, no three boundary vertices are collinear, and thus no boundary vertex is collinear with all of its neighbors.

For the sake of argument, suppose some vertex \( u \) and all of its neighbors lies on a common line \( \ell \), which without loss of generality is horizontal. Let \( V^+ \) and \( V^- \) be the subsets of vertices above and below \( \ell \), respectively. Let \( U \) be the set of all vertices that are reachable from \( u \) and whose neighbors all lie on \( \ell \). The halfplane lemma implies that the induced subgraphs \( G[V^+] \) and \( G[V^-] \) are connected, and the induced subgraph \( G[U] \) is connected by definition. Fix arbitrary vertices \( v^+ \in V^+ \) and \( v^- \in V^- \).

Finally, let \( W \) denote the set of all vertices that lie on line \( \ell \) and are adjacent to vertices in \( U \), but are not in \( U \) themselves. Every vertex in \( W \) has at least one neighbor not in \( \ell \), so by the both-sides lemma, every vertex in \( W \) has neighbors in both \( V^+ \) and \( V^- \). Deleting the vertices in \( W \) disconnects \( U \) from the rest of the graph. Thus, because \( G \) is 3-connected, \( W \) contains at least three vertices \( w_1, w_2, w_3 \).

Now suppose we contract the induced subgraphs \( G[V^+], G[V^-], \) and \( G[U] \) to the vertices \( v^+, v^-, \) and \( u \), respectively. The resulting minor of \( G \) contains the complete bipartite graph \( \{v^+, v^-, u\} \times \{w_1, w_2, w_3\} = K_{3,3} \). But this is impossible, because \( G \) is planar and therefore every minor of \( G \) is planar. \( \square \)

![Figure 2: A collinear vertex neighborhood implies a \( K_{3,3} \) minor.](image)

**Both sides redux:** Every interior vertex \( v \) has neighbors on both sides of any line through \( p_v \).

### 12.4 No Degenerate Faces

It remains to prove that the faces of the Tutte drawing are nondegenerate. First we need a combinatorial lemma.

**Geelen's Lemma:** Let \( uv \) be any edge of \( G \), let \( f \) and \( f' \) be the faces incident to \( uv \), and let \( S \) and \( S' \) be the vertices of these two faces other than \( u \) and \( v \). Let \( P \) be any path that starts at a vertex in \( S \) and ends at a vertex of \( S' \). Then every path from \( u \) to \( v \) in \( G \) either consists of the edge \( uv \) or contains a vertex of \( P \).
Proof: Fix any planar embedding of $G$ (not necessarily the Tutte drawing!) where $uv$ is an interior edge. The faces incident to $uv$ are disjoint disks on either side of $uv$. Let $s$ and $t$ be the endpoints of $P$. Let $P'$ be a path from $l$ to $r$ through the union of the faces incident to $uv$, crossing the edge $uv$ once. The closed curve $C = P + P'$ separates $u$ from $v$. Thus, by the Jordan curve theorem, every path $Q$ from $u$ to $v$ crosses $C$, which implies that either $Q = uv$ or $Q$ contains a vertex of $P$. □

Figure 3: Geelen’s lemma.

Split Faces Lemma: Let $uv$ be any interior edge of $G$, let $f$ and $f'$ be the faces incident to $uv$, and let $S$ and $S'$ be the vertices of these two faces other than $u$ and $v$. Finally, let $\ell$ be any line through $p_u$ and $p_v$. Then $S$ and $S'$ lie on opposite sides of $\ell$; in particular, no vertex in $S \cup S'$ lies on $\ell$.

Proof: Without loss of generality, assume $\ell$ is horizontal. For the sake of argument, suppose both $S$ and $S'$ contain vertices $s$ and $t$ that lie on or below $\ell$. If $s$ lies on $\ell$, the nondegeneracy lemma implies that $s$ has a neighbor $s'$ strictly below $\ell$; otherwise, let $s' = s$. Similarly, if $t$ lies on $\ell$, the nondegeneracy lemma implies that $t$ has a neighbor $t'$ strictly below $\ell$; otherwise, let $t' = t$. The halfspace lemma implies that there is a path $P'$ in $G$ from $s'$ to $t'$ that lies entirely below $\ell$. Let $P$ be the path from $s$ to $t$ consisting of the edge $ss'$ (if $s \neq s'$), the path $P'$, and the edge $t't'$ (if $t \neq t'$).

The nondegeneracy lemma also implies that $u$ and $v$ have respective neighbors $u'$ and $v'$ strictly above $\ell$, and the halfspace lemma implies that there is a path $Q'$ from $u'$ to $v'$ that lies strictly above $\ell$. Let $Q$ be the path from $u$ to $v$ consisting of the edge $uu'$, the path $Q'$, and the edge $v'v$.

The edge $uv$ and the path $P$ satisfy the conditions of Geelen’s lemma. The path $Q$ clearly avoids the edge $uv$, so $Q$ must cross $P$. But $P$ and $Q$ lie on opposite sides of $\ell$. We have reached a contradiction, completing the proof. □

Corollary: No edge of $G$ maps to a single point.

Proof: Suppose $p_u = p_v$ for some edge $uv$. Let $\ell$ be any line through $p_u = p_v$ and some other vertex on a face incident to $uv$. We immediately have a contradiction to the previous lemma. □

Convexity lemma: Every face of $G$ maps to a strictly convex polygon.

Proof: Let $f$ be any face of $G$, let $uv$ be any edge of $f$, and let $\ell$ be the unique line containing $p_u$ and $p_v$. If $uv$ is a boundary edge, the outer face lemma implies that every vertex of $f$ except $u$ and $v$ lies strictly on one side of $\ell$. Similarly, if $uv$ is an interior edge, the split faces lemma implies that every vertex of $f$ except $u$ and $v$ lies strictly on one side of $\ell$. 5
In particular, no other vertex of \( f \) lies on the line \( \ell \). It follows that \( uv \) is an edge of the convex hull of \( f \). We conclude that \( f \) coincides with its convex hull. □

Now we are finally ready to prove the main theorem.

**Proof:** Call a point *generic* if it does not lie in the image of the Tutte drawing. Consider any path from a generic point \( p \) out to infinity that does not pass through any vertex in the drawing. The split faces lemma implies that whenever the moving point crosses an edge \( e \), it leaves one face and enters another. When the moving point reaches infinity, it is only in the outer face. Thus, every generic point lies in exactly one face.

For the sake of argument, suppose two edges \( uv \) and \( xy \) intersect in the Tutte drawing. Then any generic point near the intersection \( uv \cap xy \) must lie in two different faces, which we just showed is impossible. We conclude that the Tutte drawing is an embedding; in particular, every face is a simple polygon. We already proved that every face in this embedding is strictly convex. □

### 12.5 Not Appearing

- Weakly convex faces and internal 3-connectivity
- Colin de Verdière matrices and spherical spectral embeddings
- More spectral graph algorithms!