Generic closed curve
\( \gamma : S^1 \to \mathbb{R}^2 \)
- Simple (injective) is boring
- Self-intersections: \( \gamma(t) = \gamma(t') \)
- No triple intersections
- Every self-intersection is transverse

For some \( \varepsilon \)
\[ \gamma((t-\varepsilon, t+\varepsilon)) \]
and \( \gamma((t'-\varepsilon, t'+\varepsilon)) \)
is homeomorphic to two orthogonal lines.

No:
\[ \times \times \times \times \]

"Generic"
Closed curve
Polygon
Generic polygon.

Compactness argument \( \Rightarrow \) Finite \# crossing S.

Isotopy
1. Homotopy that preserves genericity or \# self-intersection pts
2. (Ambient isotopy)
   \[ H : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2 \]
   \[ H(0, \cdot) = \text{id} \]
   \[ H(t, \cdot) \text{ is a homeomorphism} \]

\( \Rightarrow \)
\( \Rightarrow \)
Image graph of curve

4-regular plane graph
vertices = self int pts
edges = curve segment

Not necessarily simple
- loops ok
- parallel edges ok

Multicurve
\[ \searrow \text{constituent curve} \]

Image graph
4-regular connected \( \Rightarrow \) Eulerian
Gaussian closed walk "goes straight through vertices"
Curve \( \Rightarrow \) Gauss is Eulerian

Homotopy moves
\[ \begin{array}{c}
\text{monogon} \\
\text{bilon} \\
\text{triangle} \\
\text{neither mono nor bi}
\end{array} \]

Reidemeister moves
Any two generic closed curves are connected by a finite sequence of homotopy moves $O(n^2)
abla$

Idea:

- Spindle
- Irreducible spindle
- Loop
- Monogon

$\leq n$ 3 moves empties spindle $\triangleright$ bigon

$\leq n$ 2 $\triangleright$ 0 or 1 $\triangleright$ 0 moves makes curve simple

$\Theta(n^{3/2})$ [E-Chang]

Lemma: Planar curve with $n$ vertices has $n+2$ faces.

Base: $\circ$ $n=0$ $f=2$

$\ell \rightarrow \bigcirc$

$n'=n-1$

$F'=F-1$

So $f-n$ is preserved $\Rightarrow Z$

Euler's formula $\nabla v-E+f=2$
signed Gauss codes

Label vertices
Record sequence of vertices hit by point moving around curve

abcdefgchaig...

Unsigned

Abcdefgchaig...

Signed double permutation

ABab

f = 2
v = 2

Thm: signed dbl perm is signed Gauss code of planar curve iff \( f = n + 2 \)

Next time: unsigned Gauss codes.