Euler's Formula \[ V - E + F = \Omega \]

**Deletion**

\[ G \quad \rightarrow \quad G \setminus e \]

- subgraph
- bridge

**Contraction**

\[ G \]

\[ e \]

- minor
- loop

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**while** \( E \neq \emptyset \)

\[ e \Leftarrow \text{any edge} \]

- if \( e \) is not bridge
  - delete \( e \)
- if \( e \) is not loop
  - contract \( e \)

**Lemma:**

- \( e \) is a loop \( \Rightarrow \) \( e \) not in any sp. tree
- \( e \) is a bridge \( \Rightarrow \) \( e \) in every sp. tree
- \( e \) is not a loop \( \Rightarrow \) \( T = \text{sp. tree of } G/e \)
  \( \Rightarrow T + e \) is sp. tree of \( G \)
- \( e \) is not a bridge, \( T = \text{sp. tree of } G \setminus e \)
  \( \Rightarrow T \) is sp. tree of \( G \)

**Lemma:**

- Every cycle has \( \geq 1 \) blue edge
  - Every cut crossed by \( \geq 1 \) red edge
  - \( \Rightarrow \) blue = sp. tree.

**Tarjan's red-blue rule:**

Color \( E \) red/blue so that

\[ \Rightarrow \text{blue} = \text{sp. tree.} \]
Planar maps

\[(\Sigma \setminus e)^* = \Sigma^* / e^*\]
\[(\Sigma / e)^* = \Sigma^* \setminus e^*\]

contraction is dual to deletion

\[(\text{succ } e)(d) = \begin{cases} 
\text{succ\{succ\{succ\{d\}\}\}} & \text{if succ\{d\} \in e and s\{s\{d\}\} \in e} \\
\text{succ\{succ\{d\}\}} & \text{if succ\{d\} \in e} \\
\text{succ\{d\}} & \text{o/w}
\end{cases}\]

\[(\text{succ}^* / e)(d) = \begin{cases} 
\text{s}^* (\text{s}^* (\text{s}^* \{d\})) & \text{o/w} \\
\text{succ}^* \{\text{succ}^* \{d\}\} \quad & \text{if succ\{d\} \in e} \\
\text{succ}^* \{d\} & \text{o/w}
\end{cases}\]

Medial map $\Sigma^*$
Whitney [1936]

even subgraphs / multieves

boundary between white faces + black faces

cycle space via $\Theta$

cycle $= \text{minimal nonempty even subgraph}$

edge cut

"boundary" between white vertices + black vertices

cut space (via $\Theta$)

bond $= \text{minimal nonempty edge cut}$

$\times(\ast G) = \times^2$
Tree-cotree decomposition

Planar map $\Xi = (V, E, F)$

Partition $E = T \cup C$

s.t. $T =$ sp tree of $\Xi$

$\iff C^* =$ sp. tree of $\Xi^*$

Euler: any planar map: $V - E + F = 2$

Proof 1: Induction

- no edges $\implies V = 1$ $F = 1$ $\iff (\emptyset, \emptyset, \emptyset)$

- or pick any edge $e$
  - not loop, consider $\Xi / e$
    $V' = V - 1$ $E' = E - 1$ $F' = F$
  - not bridge consider $\Xi \setminus e$
    $V' = V$ $E^* = E - 1$ $F' = F - 1$

Proof 2: Von Staudt 1847

Pick any sp tree $T$ $\implies V - 1$ edges
complementary dual

$(E \setminus T)^*$ is dual sp tree $\implies F - 1$ edges

\[ \square \]
Proof 3: Look at medial map $\Sigma^X$.

Multicurve $V^X = E$
$E^X = 2E$
$F^X = V + F$

Planar $\Rightarrow F^X = V^X + 2$  \[\Box\]

**Combinatorial Gauss-Bonnet**

\[\int \kappa \, dA = 4\pi \leq \text{surface area of unit sphere}\]
\[= 2\pi \cdot 2\]
\[\Rightarrow \int \kappa \, dA = 2\pi \cdot \chi\]

Assign "exterior angle" $\angle_c$ to every corner of $\Sigma$

"curvature" \[\chi(f) = 1 - \sum_{c \in f} \angle_c\]

\[\chi(v) = 1 - \sum_{c \in v} \left(\frac{1}{2} - \angle_c\right)\]

\[\sum_v \chi(v) + \sum_f \chi(f) = 2\]

$\chi(v) = 0$
$\chi(f) = \chi \geq 2$ if $f$ is bounded
$\chi(f) = \chi \leq 2$ if $f$ is outer
Descartes' angle defect theorem

Convex polyhedron

\[ \angle C \text{ is assigned geometrically from faces} \]

\[ \Rightarrow \kappa(f) = 0 \]

\[ \kappa(u) = \text{angle defect at } u \]

\[ \sum \kappa(u) = 2 \]