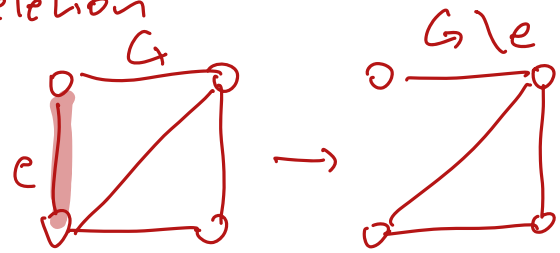
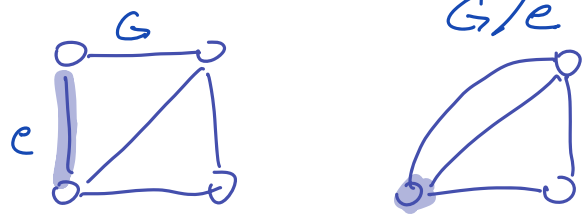
Euler's Formula $\boxed{V - E + F = 2}$

Deletion

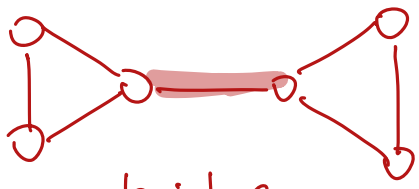


subgraph

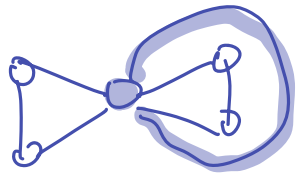
Contraction



minor



bridge



loop

---

while $E \neq \emptyset$

  $e \leftarrow$ any edge

  if e is not bridge
    delete e
  if e is not loop
    contract e

  } either order

Lemma: Contracted edges are a spanning tree of G

Lemma:

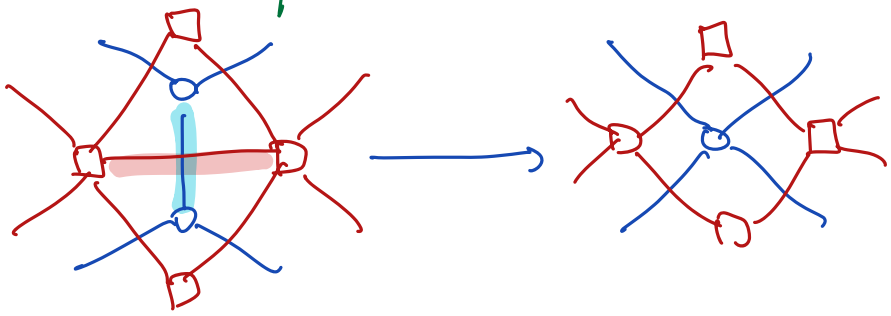
- e is a loop $\Rightarrow$ e not in any sp. tree
- e is a bridge $\Rightarrow$ e in every sp. tree
- e is <u>not</u> a loop $\Rightarrow$ T = sp tree of G/e
  $\Rightarrow$ T+e is sp tree of G
- e is is <u>not</u> a bridge, T = sp tree of G\e
  $\Rightarrow$ T is sp tree of G

Tarjan's red-blue rule:

Color E red/blue so that
- every cycle has ≥ 1 blue edge
- every cut crossed by ≥ 1 red edge

$\Rightarrow$ blue = sp. tree.

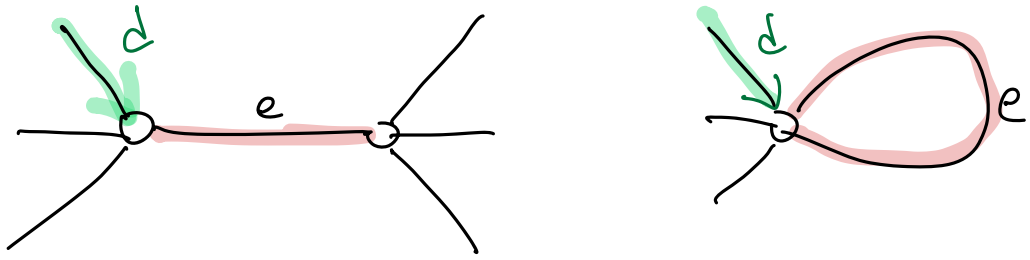
Planar maps



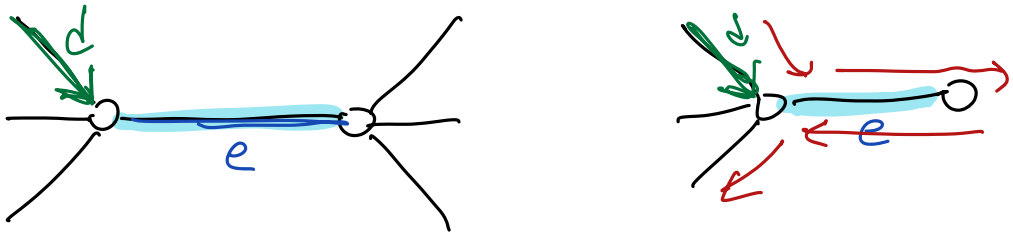
$$(\Sigma \setminus e)^* = \Sigma^* / e^*$$

$$(\Sigma / e)^* = \Sigma^* \setminus e^*$$

$$\boxed{\text{contraction is dual to deletion}}$$

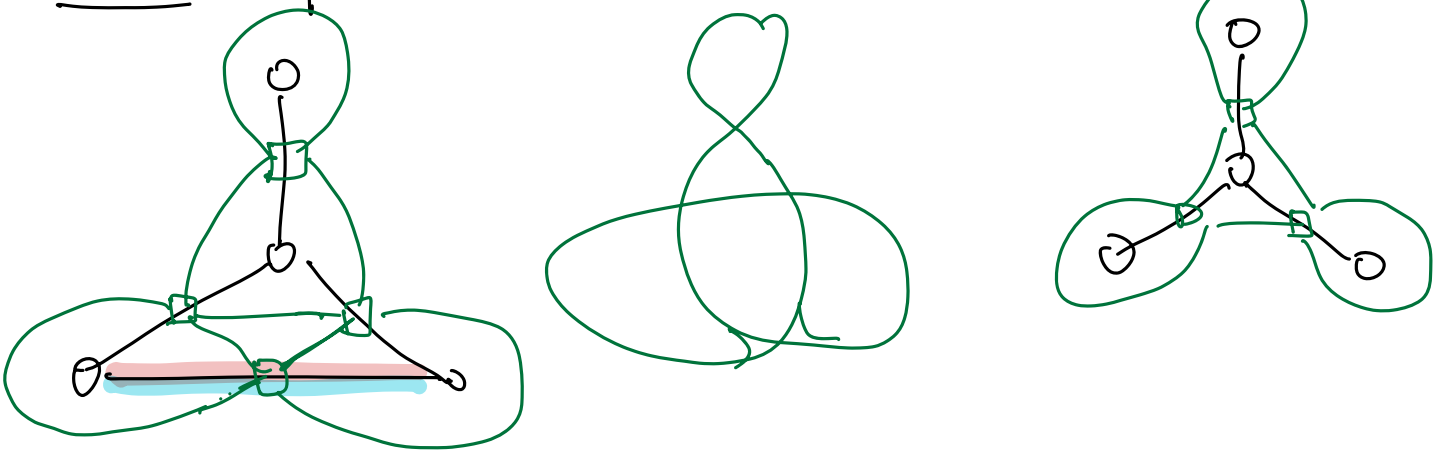


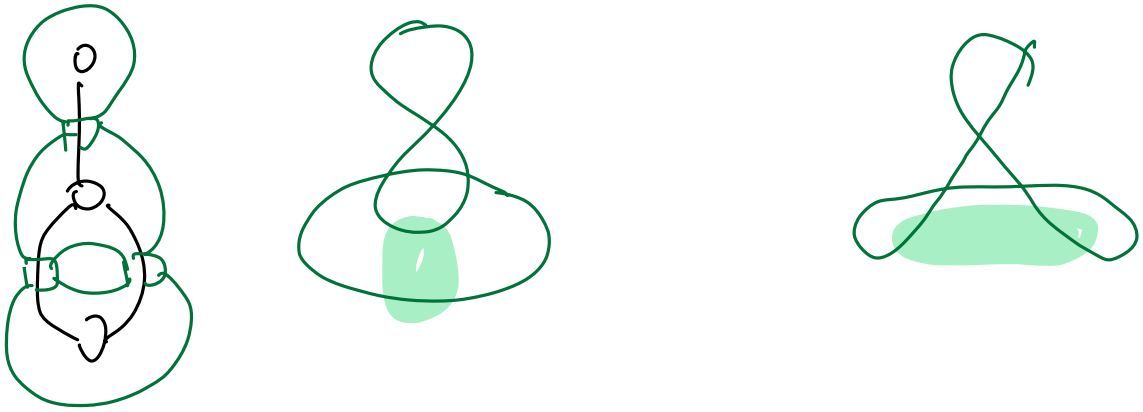
$$(\mathrm{succ} \setminus e)(d) = \begin{cases} \mathrm{succ}(\mathrm{succ}(\mathrm{succ}(d))) & \text{if } \mathrm{succ}(d) \in e \text{ and } s(s(d)) \in e \\ \mathrm{succ}(\mathrm{succ}(d)) & \text{if } \mathrm{succ}(d) \in e \\ \mathrm{succ}(d) & \text{o/w} \end{cases}$$

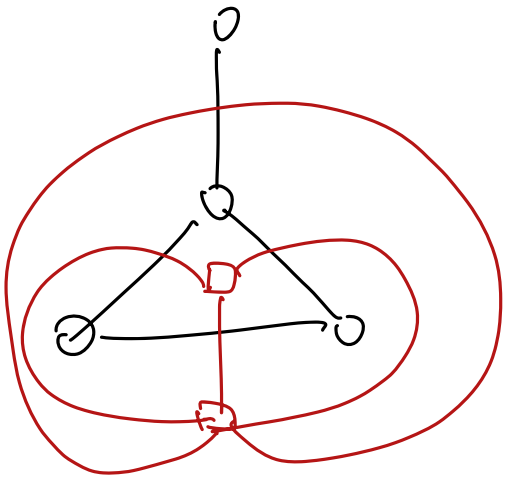


$$(\mathrm{succ}^* / e)(d) = \begin{cases} s^*(s^*(s^*(d))) \\ \mathrm{succ}^*(\mathrm{succ}^*(d)) \\ \mathrm{succ}^*(d) \end{cases}$$

Medial map $\Sigma^\times$







$$\Sigma^{\times} = (\Sigma^{*})^{\times}$$

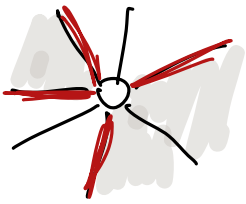
Whitney [1936]

even subgraphs / multicurves

= boundary between white faces + black faces

↓

cycle space via ⊕



cycle = minimal nonempty even subgraph

edge cut

= "boundary" between white vertices + black vertices

↓

cut space (via ⊕)

bond = minimal nonempty edge cut

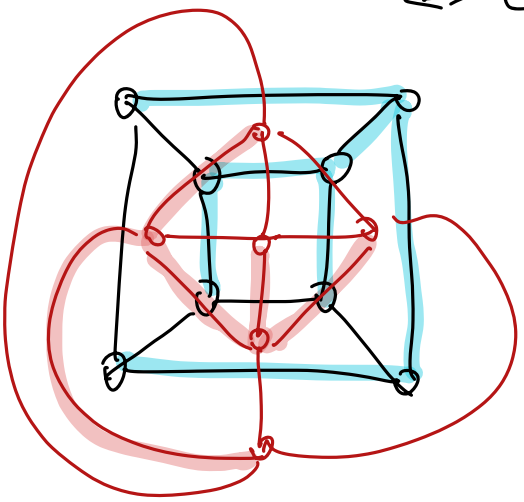
Tree-cotree decomposition

Planar map $\Sigma = (V, E, F)$

$\hookrightarrow$ Partition $E = T \sqcup C$

s.t. T = sp tree of Σ

$\Leftrightarrow C^* =$ sp. tree of Σ^*



medial $\Rightarrow$ ~~tree~~ coonion-onion diagrams

Euler: any planar map: $V - E + F = 2$

Proof 1: Induction

- no edges $\quad V = 1 \quad F = 1 \qquad \bullet \qquad (\emptyset, \emptyset, \emptyset)$
- o/w pick any edge e
  - not loop, consider Σ / e
    $V' = V - 1 \quad E' = E - 1 \quad F' = F$
  - not bridge consider $\Sigma \setminus e$
    $V' = V \quad E' = E - 1 \quad F' = F - 1 \qquad \square$

Proof 2: Von Staudt 1847

Pick any sp tree T $\longrightarrow$ $V - 1$ edges

complementary dual

$(E \setminus T)^*$ is dual sp tree $\longrightarrow$ $F - 1$ edges

$\square$

Proof 3: Look at medial map $\Sigma^\times$

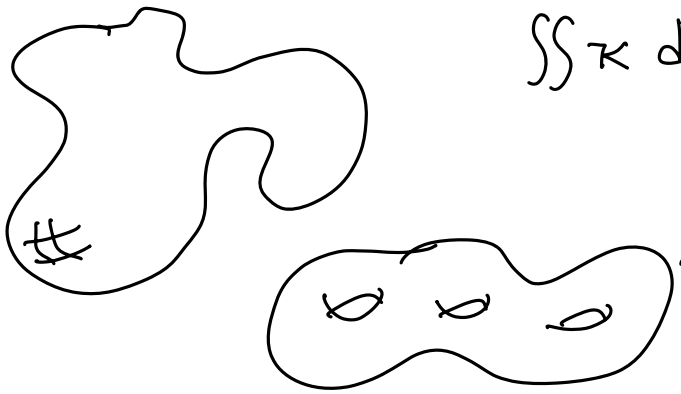
Multicurve $V^\times = E$

$E^\times = 2E$

$F^\times = V + F$

Planar $\Rightarrow F^\times = V^\times + 2$ $\square$

Combinatorial Gauss-Bonnet



$$\iint \kappa \, dA = 4\pi$$ surf area of unit sphere

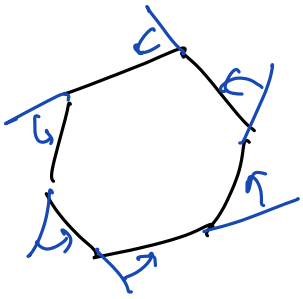
$$= 2\pi \cdot 2$$

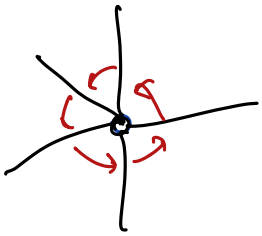
$$\iint \kappa \, dA = 2\pi \cdot \chi$$

Assign "exterior angle" $\angle c$ to every corner of Σ

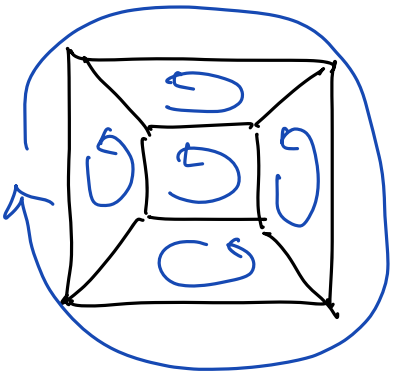
"curvature" $\kappa(f) = 1 - \sum_{c \in f} \angle c$

$\kappa(v) = 1 - \sum_{c \in v} \left(\frac{1}{2} - \angle c\right)$





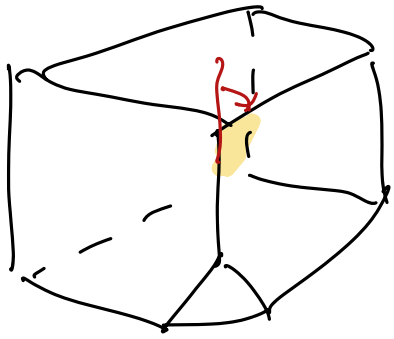
$$\boxed{\sum_v \kappa(v) + \sum_f \kappa(f) = 2}$$



$\kappa(v) = 0$

$\kappa(f) = \begin{cases} 0 & \text{if } f \text{ is bounded} \\ 2 & \text{if } f \text{ is outer} \end{cases}$

Descartes' angle defect theorem



convex polyhedron

$\angle c$ is assigned geometrically from faces

$\Rightarrow \kappa(f) = 0$

$\kappa(v) =$ angle defect at v

$$\boxed{\sum \kappa(v) = 2}$$