W. Tutte "How to draw a graph" (1963)

Given: Planar graph $G$ 3-connected

Face $f_0$ of some embedding of $G$ in plane

Convex polygon $P$ with $|f_0|$ vertices

Output: Tutte drawing $p : V(G) \rightarrow \mathbb{R}^2$

- for each vertex $v_i$ on $f_0$, $p(v_i) = P_i$
- any vertex $v$ not on $f_0$

\[ \sum_{uv} \lambda_{uv} (p_u - p_v) = 0 \]

For some $\lambda_{uv} > 0$ can be given

If $\lambda_{uv} = \lambda_{vu} = w_{uv}$

potential energy $\Phi(p) = \sum_{uv} w_{uv} |p_u - p_v|^2$

$\nabla \Phi = 0$

Theorem: Every Tutte drawing of 3-connected simple planar graph is a convex embedding
Outer Face is outer: For every vertex not on \( \partial P \), \( v \) is in interior of \( P \)
\[ v = \text{all of } v' \text{'s nbs} \]

Star mesh transformation at \( v \)

3-conn \( \Rightarrow v \) can directly reach at least 3 vertices of \( \partial P \)
even after pivoting \( v \) out

Lemma:
If \( v \) has nbr on one side of line \( l \) then \( P_v \) then other side too.

Halfplane lemma

For any halfplane \( H \) that intersects \( P \), subgraph of \( G \) induced by vertices in \( H \) is connected

There is a path from \( v \) to \( t \) where \( y \)-coords never decrease

1. \( y_t = y_v \)
2. \( U = \text{all verts reachable from } v \text{ via horiz-edges} \)
   \( w = \text{some vert } x \text{ in } U \text{ with non-horiz nbs} \)
   \( w \rightarrow x \text{ with } x \text{ above } w \)
   \( \text{In } : x \rightarrow t \)
No flat vertex neighborhoods

Utility Lemma: $K_{5,3}$ isn't planar
$\iff$ Euler

1. No outer vertex $v$

$U = \text{reachable from } u$ and all nbrs on $l$

$3$-conn $\Rightarrow$
At least $3$ disjoint paths from $u$ to boundary

$w_1, w_2, w_3 \times U, V^+, V^-$
gives subdivision of $K_{5,3}$
$K_{5,3}$ is minor of $G$
$\Rightarrow G$ non-planar

No degenerate faces

Gecen's Lemma:

Let $P$ = any path from $S$ to $S'$

Every path from $u$ to $v$ except $uv$ crosses $P$

$JCT \Rightarrow$ Lemma $\Box$

[whitney]: 3-con planar graph has unique
embedding (up to homeo) on sphere
Split Faces: Let $uv$ = any interior edge
$f, f' = $ faces incident to $uv$
$s, s' = $ verts of $f, f'$ not $u$ or $v$
$l = $ any line thru $p$, and $pu$
$s$ and $s'$ lie on opposite sides of $l$ not on $l$

$\Rightarrow pu \neq pv$

$uv, P, and Q$ satisfy conditions for Geelen's Lemma
$\Rightarrow P$ and $Q$ intersect
But $Q \perp l$ and $P \parallel l$ !

No face is flat
$\Rightarrow$ Every face is strictly convex

Last step:
Any pt.$ \in P$

is inside $\leq 1$ Face of Tutte-drawing

Faces can't overlap $\Rightarrow$ embedding