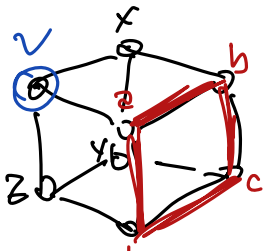
Paper Chase due next Tue Feb 28

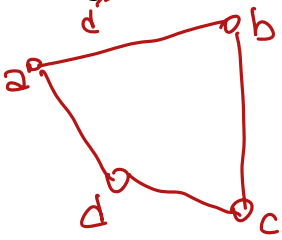
W. Tutte "How to draw a graph" (1963)

Given: Planar graph G — 3-connected

Face f_∞ of some embedding of G in plane

Convex polygon P with $|f_\infty|$ vertices





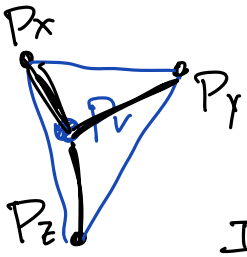
Output: Tutte drawing $p: V(G) \to \mathbb{R}^2$

- for each vertex v_i on f_∞ $\quad p(v_i) = p_i$ ← vertex of P

- Every vertex v not on f_∞

$$\boxed{\sum_{u \to v} \lambda_{u \to v} (p_v - p_u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

For $\boxed{\text{some } \lambda_{uv} > 0}$ ← can be given

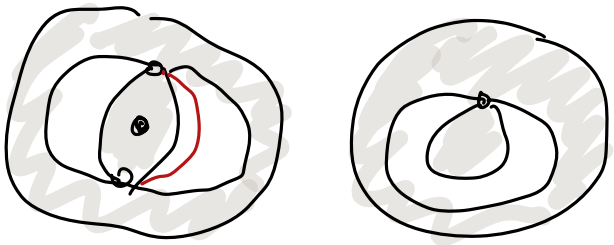


If $\lambda_{u \to v} = \lambda_{v \to u} = \omega_{uv}$

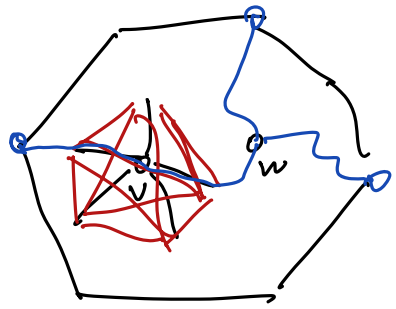
potential energy $\Phi(p) = \sum_{uv} \omega_{uv} |p_u p_v|^2$ ← convex

$\nabla \Phi = 0$

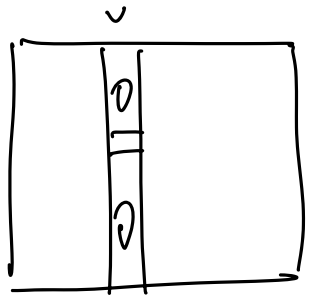
Theorem: Every Tutte drawing of 3-conn simple planar graph is a convex embedding (necessary)



Outer face is outer : For every vertex w not on f_∞ P_w is in interior of P



P_v = wt ave of v's nbrs

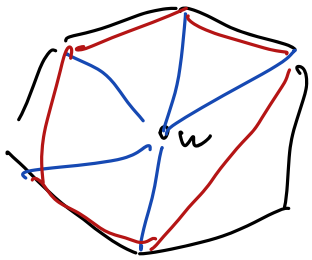


star mesh transformation at v

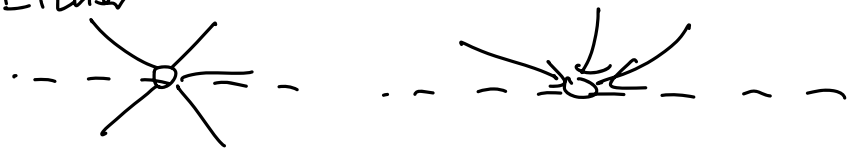
Y-Δ

3-conn $\Rightarrow$ w can directly reach at least 3 vertices of f_∞

even after pivoting v out



Either

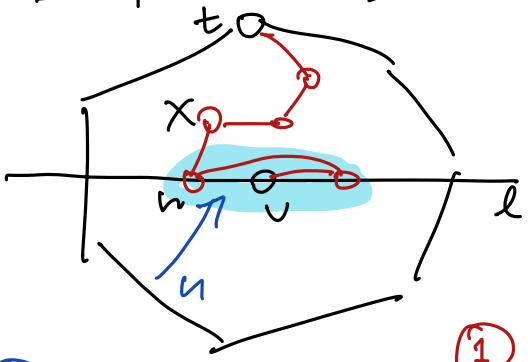


or

~~Lemma:~~
If v has nbr on one side of line ℓ thru P_v then other side too.

Halfplane lemma

For any halfplane H that bdry intersects P subgraph of G induced by vertices in H is connected



There is a path from v to t where y-coords never decrease

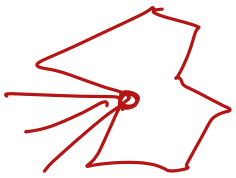
① $y_t = y_v$ ✓

② U = all verts reachable from v via horiz. edges

w = some vertex in U with non-horiz nbrs

$w \to x$ with x above w

Ind: $x \rightsquigarrow t$ □

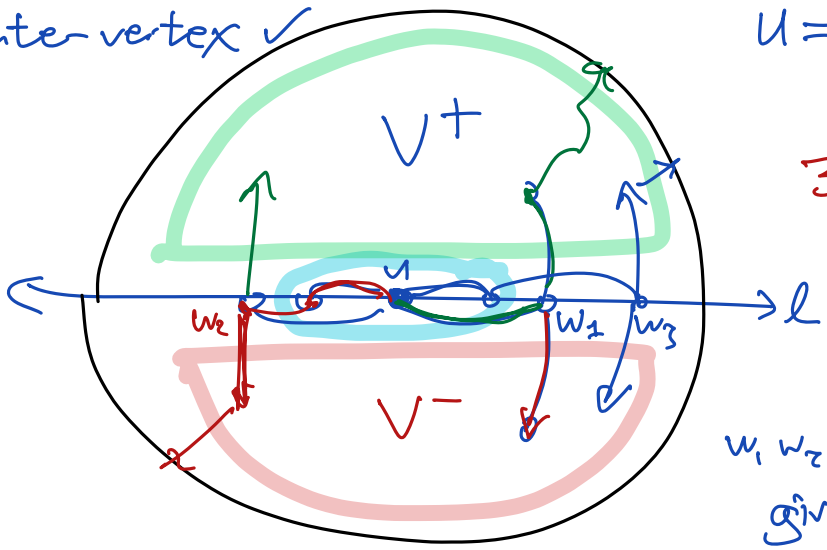


No flat vertex neighborhoods

Utility Lemma: $K_{3,3}$ ain't planar

$\Leftarrow$ Euler

① No outer vertex ✓



U = reachable from u and all nbrs on ℓ

3-conn $\Rightarrow$ At least 3 disjoint paths from u to boundary

$w_1\, w_2\, w_3 \times U, V^+, V^-$ gives subdivision of $K_{3,3}$

$K_{3,3}$ is minor of G

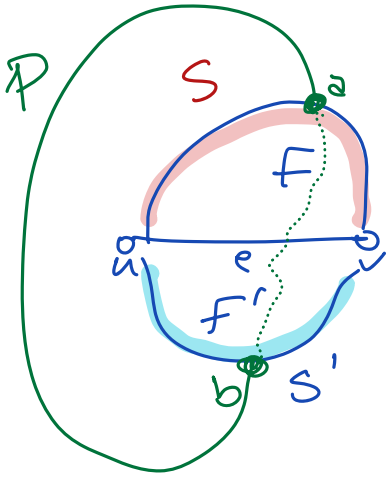
$\Rightarrow$ G non planar ⊠

No degenerate faces

Geelen's Lemma:

Let P = any path from s to s'

Every path from u to v except uv crosses P



$S = \text{vert}\, F \setminus \{u, v\}$

$S' = \text{vert}\, F' \setminus \{u, v\}$

JCT $\Rightarrow$ Lemma □

[Whitney]: 3-con planar graph has unique embedding (up to homeo) on sphere

Split Faces: Let uv = any interior edge

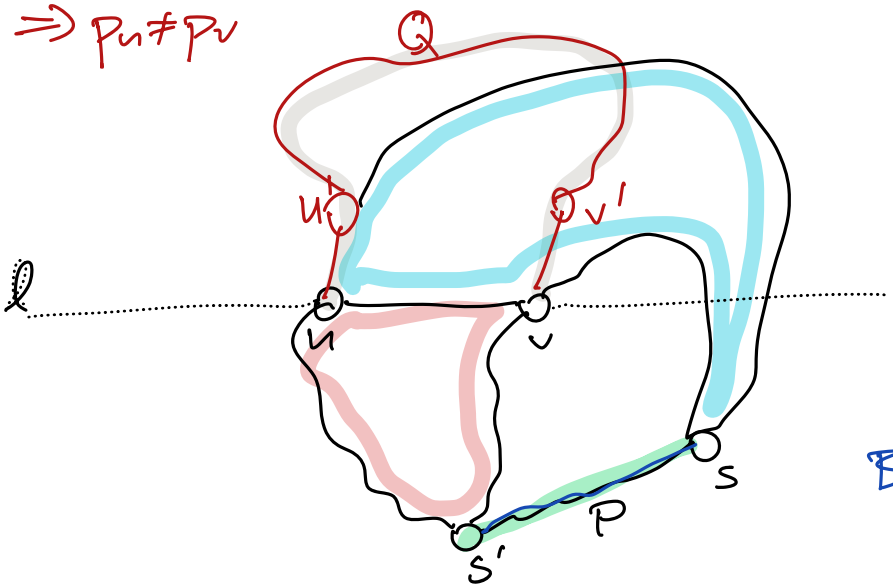
f, f' = faces incident to uv

s, s' = verts of f, f' not u or v

ℓ = any line thru p_u and p_v

s and s' lie on opposite sides of ℓ not on ℓ

$\Rightarrow p_u \neq p_v$



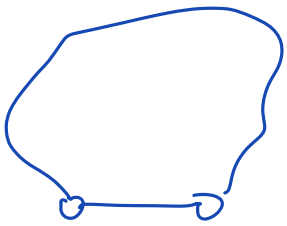
uv, P, and Q satisfy conditions for Geelen's Lemma

$\Rightarrow$ P and Q intersect

But $Q \uparrow \ell$ and $P \downarrow \ell$ ※

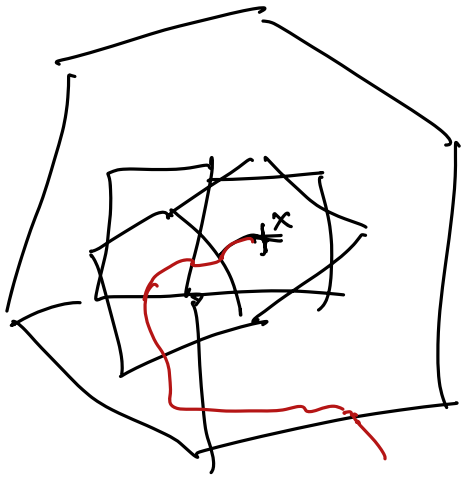
No face is flat

$\Longrightarrow$ Every face is strictly convex



Last step:

Any point inside P is inside ≤ 1 face of Tutte drawing



Faces can't overlap $\Rightarrow$ embedding

$\square$