Stevin 1605

Varignon 1725

\begin{itemize}
  \item equilibrium stress in a planar framework
  \item reciprocal framework ("force diagram")
  \item polyhedral lift
\end{itemize}

\[ G = \text{3-connected simple planar graph} \]
\[ L = \text{unique embedding} \]
\[ p = \text{position} \quad p: V(G) \to \mathbb{R}^2 \]
\[ \mathbb{P}^2(G, p) \text{ is planar Framework} \]
Stress: \( \omega: E(G) \rightarrow \mathbb{R} \)
\[ + \rightarrow \]

Equilibrium:
\[ \sum_{u} \omega(uv) \cdot (p(v) - p(u)) = 0 \]

\[ \nabla \Phi = 0 \text{ where } \]
\[ \Phi = \sum_{uv} \omega(uv) \cdot \| p(v) - p(u) \|^2 \]

discrete 1-form (pseudoflow)
\[ \phi: D(G) \rightarrow \mathbb{R} \text{ or } \mathbb{R}^2 \]
\[ \phi(d) = -\phi(\text{rev}(d)) \]

exact — sum around any cycle = 0
always planar

closed — sum around any face = 0

Lemma: \( \phi \) is exact iff \( \phi(u \rightarrow v) = \Pi(v) - \Pi(u) \)
for some \( \Pi: V(G) \rightarrow \mathbb{R}^2 \)
vertex potential

Proof: If \( \phi = \delta \Pi \)

\[ \sum_{uv} \phi(u \rightarrow v) \text{ is telescoping} \]
\[ \sum_{uv} \phi(u \rightarrow v) = \sum_{uv} (\Pi(v) - \Pi(u)) = 0 \]

If \( \phi \) exact
Assign \( \Pi(s) = 0 \)
Lemma: Equilibrium stress in \((G, p)\)

\[ \Delta = (z, o) \]

Reciprocal diagram \((G^*, p^*)\) (unique up to translation)

Proof:

\( \omega \) is eq. stress for \((G, p)\)

\[ \Delta(u \rightarrow v) = \delta p(u \rightarrow v) \]

\[ = p(v) - p(u) \]

Dual displacement \( \Delta^*(d^*) = \omega(d) \cdot \Delta(d)^\perp \)

Consider any face \( u^\star \) of \( G^* \)

\[ \sum \Delta^*(d^*) = \sum \omega(d) \cdot \Delta(d)^\perp \]  
\( d^*: \text{left}(d^*) = u^\star \)  
\( \text{head}(d) = v \)

\((G^*, p^*)\) any reciprocal diagram

\[ (\gamma^\perp) = (\gamma)^\perp = (\gamma) \]

\( \Delta^* \) is exact

There is a unique \( \omega: D(b) \to \mathbb{R} \)

s.t. \( p^*(b) - p^*(a) = \omega(d) \cdot (p(v) - p(u))^\perp \)

\[ a \quad \xrightarrow{\delta} \quad b \]

\[ u \quad \xrightarrow{\omega} \quad v \]
Polyhedral lifts of \((G,p)\) is
\[ z: V(G) \to \mathbb{R} \] "height"
\[
\hat{\beta}(v) = (x(v), y(v), z(v)) \text{ where } (x(v), y(v)) = p(v)
\]
vertices on any face of \(G\) have coplanar 3d coordinates

non-trivial: not all equal \(z\)'s
not all coplanar

Lemma: reciprocal diagram \((G^*, p^*)\)
non-trivial poly lift \(z\).

radial graph/map \(G^0\)
vertices = \(V(G) \cup V(G^*)\)
edges = corners of \(G\) (or \(G^*\))
faces = edges of \(G\) (or \(G^*\))

\[
Z = \text{non-trivial poly lift of } (G,p)
\]
For each face \(F\) of \(G\)
\[
\hat{\beta}(F) \text{ lies in plane } z = x^*(F) \cdot x + y^*(F) \cdot y - z^*(F)
\]
define \(p^*(F^*) = (x^*(F), y^*(F))\)

For every corner \((v,F)\)
\[
z(v) + z^*(F) = x(v) \cdot x^*(F) + y(v) \cdot y^*(F)
\]
\[
\hat{a} \quad \begin{array}{c}
\text{u} \\
\rightarrow \\
\text{b}
\end{array}
\]

\[ p(v) \cdot p^*(F) \]
\[ p(u) \cdot p^*(a) = z(u) + z^*(a) \]
\[ (p(u) - p(w)) \cdot (p^*(a) - p^*(b)) = 0 \]

Steinitz Thm: 3-com planar \iff 1-skeleton of convex polyhedron in \( \mathbb{R}^3 \)

If \( (G,p) \) is embedded

- interior stresses +
- boundary stresses -

polyhedral lift is convex