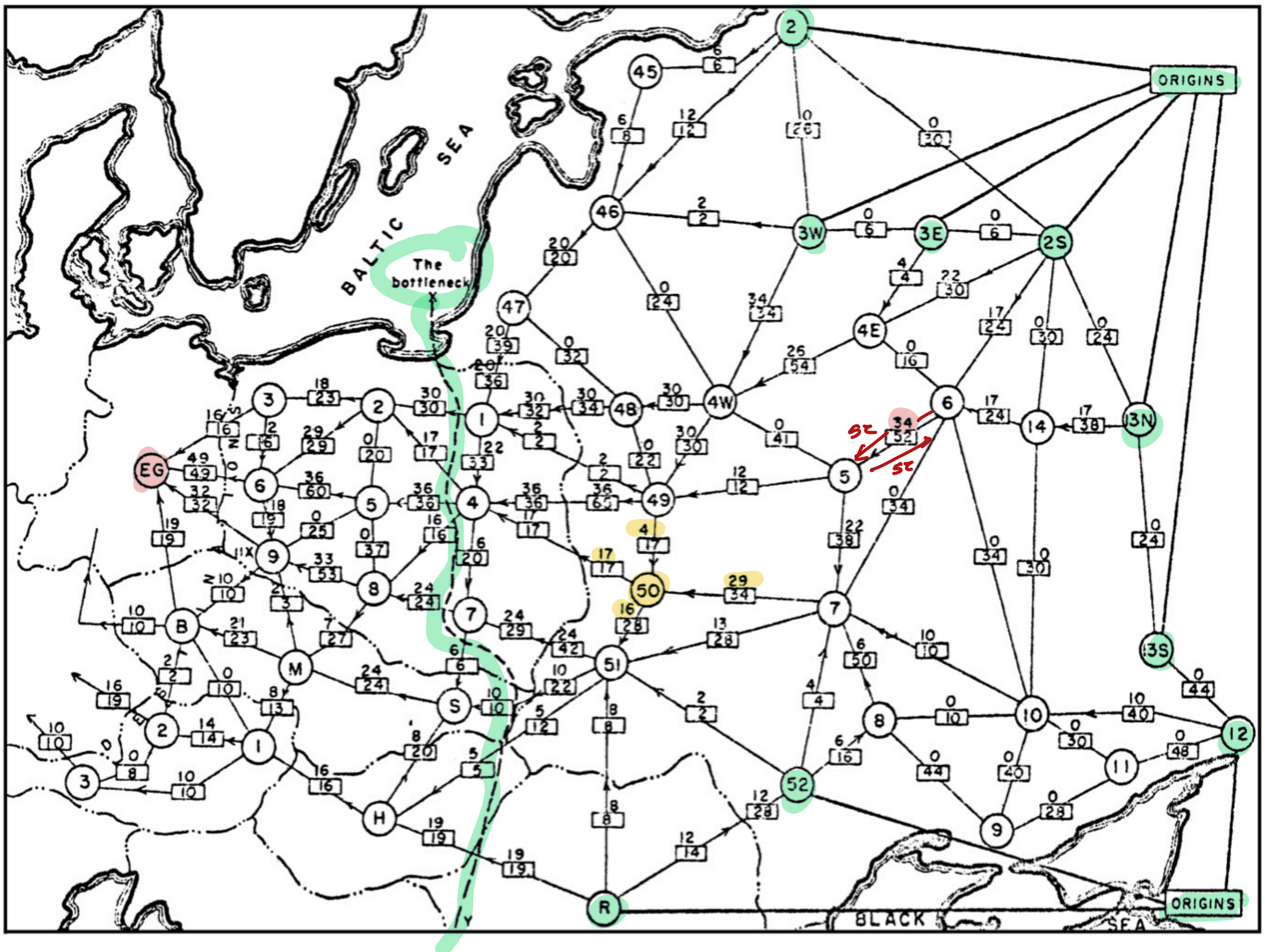
Maximum Flows



Last time:

Min (s,t)-cut in $\overbrace{\text{undirected}}^{c(u\to v)=c(v\to u)}$ planar in $O(n \log n)$ time

$\downarrow$

$O(n \log\log n)$

General graphs : $m^{1+o(1)} \cdot F(\varepsilon)$

Today: max flow in dir. planar in $O(n \log n)$ time.

Weihe '89, Borradaile Klein '06, Erickson '10

Graph G vertices s, t (maybe)

capacity $c : D(G) \to \mathbb{R}$

Pseudoflow: $\phi : D(G) \to \mathbb{R}$ $\phi(d) = -\phi(rev(d))$

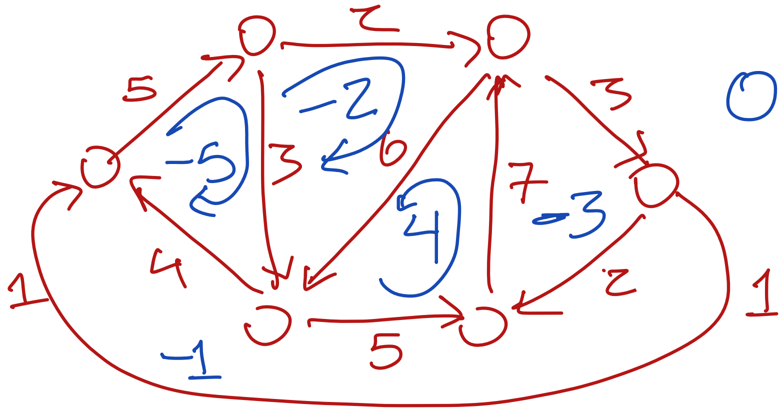
$\boxed{\text{Boundary: } \partial\phi(v) = \sum_{u \to v} \phi(u \to v)}$

Circulation: $\partial\phi(v) = 0$ for all v.

(s,t)-flow: $\partial\phi(v) = 0$ for all $v \neq s, t$

<u>value</u> $|\phi| = \partial\phi(t) = -\partial\phi(s)$

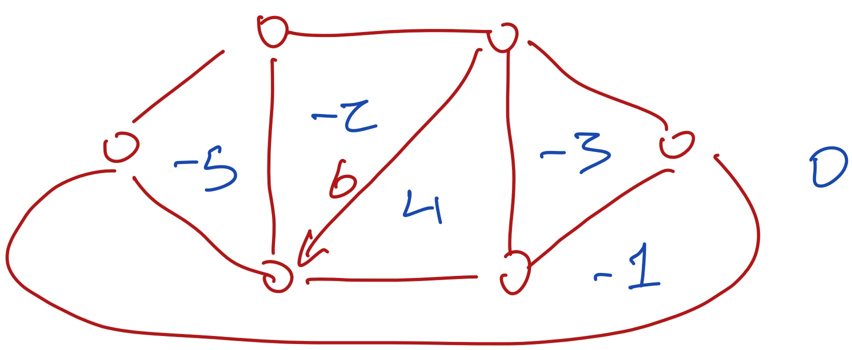
Planar



Alexander <u>numbering</u>

$\alpha(\text{outer}) = 0$

$\alpha(\text{left}(d)) = \alpha(\text{right}(d)) + \phi(d)$



2-chain

Face potential $\alpha : F(\Sigma) \to \mathbb{R}$

$\partial\alpha(d) = \alpha(\text{left}(d)) - \alpha(\text{right}(d))$

Every circulation in a planar ~~graph~~ map is a boundary circulation.

$$\phi = \sum_F \alpha_F \cdot \partial F$$

Capacities

$c : D(G) \to \mathbb{R}$

Fix an embedding Σ

Pseudoflow ϕ is feasible if $\phi(d) \le c(d)$

If $c(d) < 0$ forces $\phi(rev(d)) \ge -c(d)$

Residual network

Fix G, c, ϕ

Σ = map with given capacities c

$$\boxed{c_\phi(d) = c(d) - \phi(d)}$$

Σ_ϕ = same map with capacities c_ϕ

ϕ is feasible $\iff$ $c_\phi(d) \ge 0$ for every dart d

Ford Fulkerson:

$\phi \leftarrow 0$
repeat
  Find path π from s to t with pos. res. capacity
  $\phi \leftarrow \phi + \min_{d \in \pi} c(d)$
until no such path

ϕ, ϕ' = Flows in G

ϕ' is feasible in $G \iff \phi' - \phi$ is feasible in G_ϕ

Problem: Given planar map Σ, caps $c(d)$
Is there a feasible circulation in Σ?

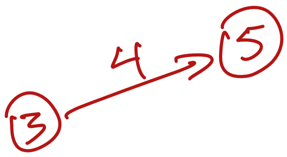
Solution: Look at Σ^* interpret $c(d)$ as cost of d^*

$O(n \log^2 n)$ time
Compute shortest path tree rooted at o — dual of outer face of Σ

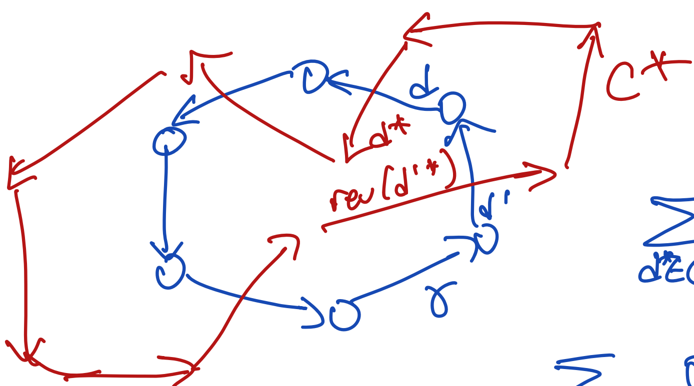
Suppose $\mathrm{dist}(f^*)$ is well-defined "$\alpha(f)$"

Claim: $\partial\alpha$ is feasible!

$$\partial\alpha(d) = \alpha(\mathrm{left}(d)) - \alpha(\mathrm{right}(d))$$
$$= \mathrm{dist}(\mathrm{left}(d)^*) - \mathrm{dist}(\mathrm{right}(d)^*)$$
$$= \mathrm{dist}(\mathrm{head}(d^*)) - \mathrm{dist}(\mathrm{tail}(d^*))$$
$$\leq c(d^*) = c(d)$$



Suppose Σ^* has neg cycle C^*
Fix any circulation ϕ wlog one ccw cycle γ



$$\sum_{d^* \in C^*} \gamma(d) = 0$$

$$\sum_{d^* \in C^*} \phi(d) = 0$$

but $\displaystyle\sum_{d^* \in C^*} c(d) < 0$

so $\phi(d) < c(d)$ for some $d^* \in C^*$ $\square$

Feasible flow:-

Given Σ, c, Flow ϕ

Is there a feasible flow ϕ' s.t. $|\phi| = |\phi'|$?

Is there a feasible circulation in Σ_ϕ ?

Max Flow

cap $c(d) \in \mathbb{N}$ $\max_d c(d) = U$

Binary search over all possible flow values

$\boxed{O(n \log^2 n \log U) \text{ time}}$