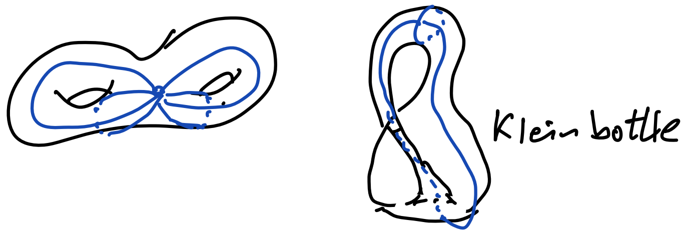
Surfaces



Surface Classification Theorem:

Kerekjarto / Rado ~ Any surface can be triangulated.

Moise — 3-manifold

Freedman — False for 4-manifolds

Fix a map $\Sigma = (V, E, F)$ of your favorite surface S,

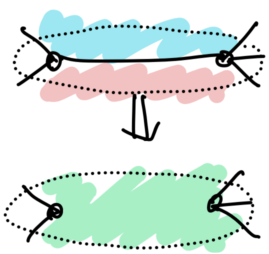
represented by reflection system (Φ, a, b, c)

band decomposition $\Sigma^{\square}$

Deletion

(two face differ)

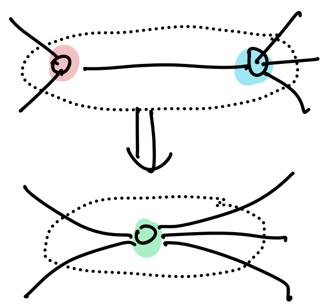
not an isthmus



Contraction

two ends differ

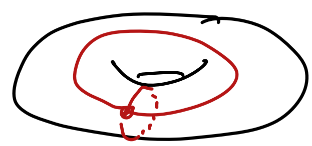
(not a loop.)



Contract spanning tree T

Delete spanning cotree C

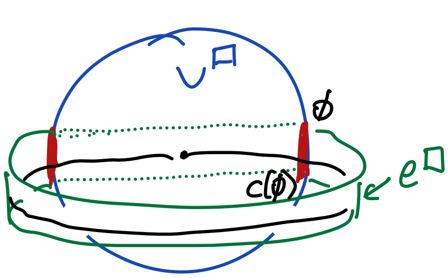
Leftover edges L



(T, L, C) — tree-cotree decomposition [Eppstein]

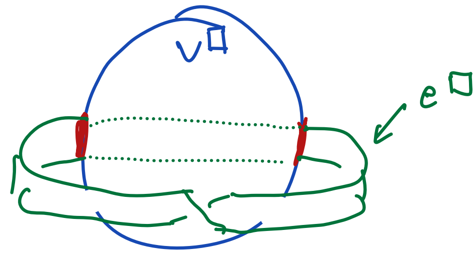
$\Sigma / T \setminus C$ ← map with one vertex v
and one face f

System of loops $e_1, e_2, \ldots, e_L$



two sided loop

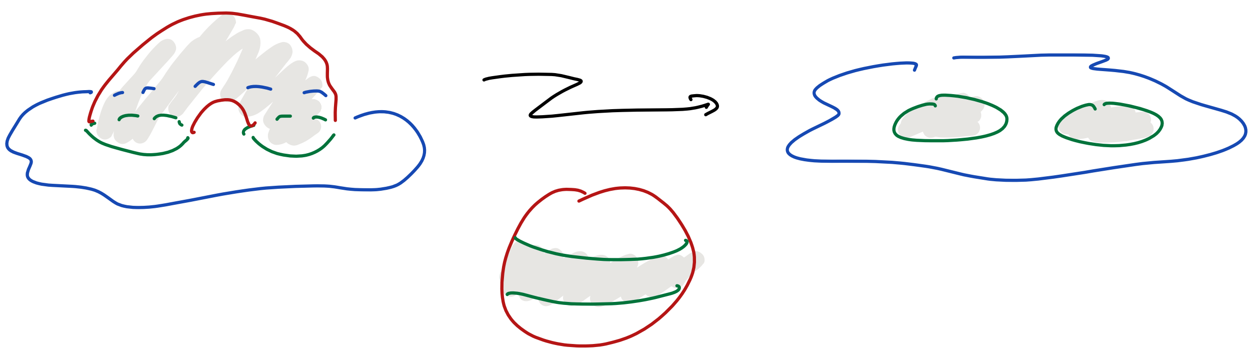
~~annulus~~
"handle"



one-sided loop

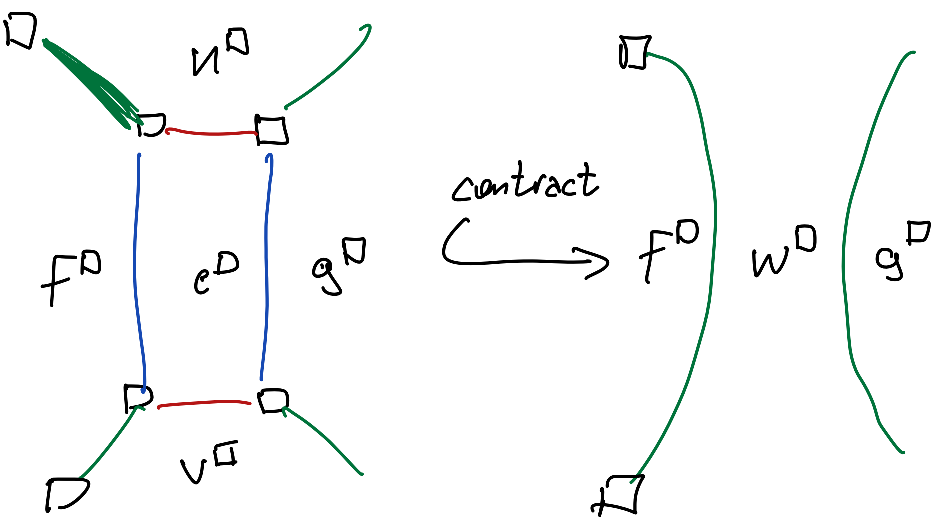
möbius band
"twist"

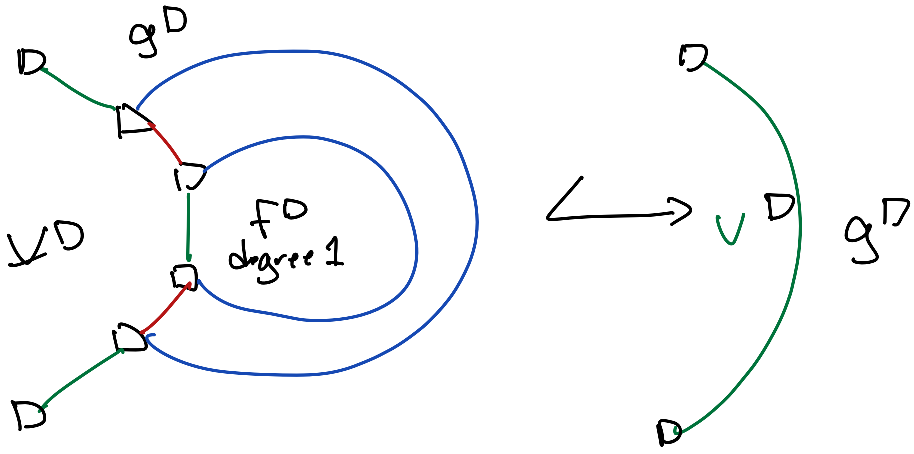
Σ is Orientable iff every loop is 2-sided.

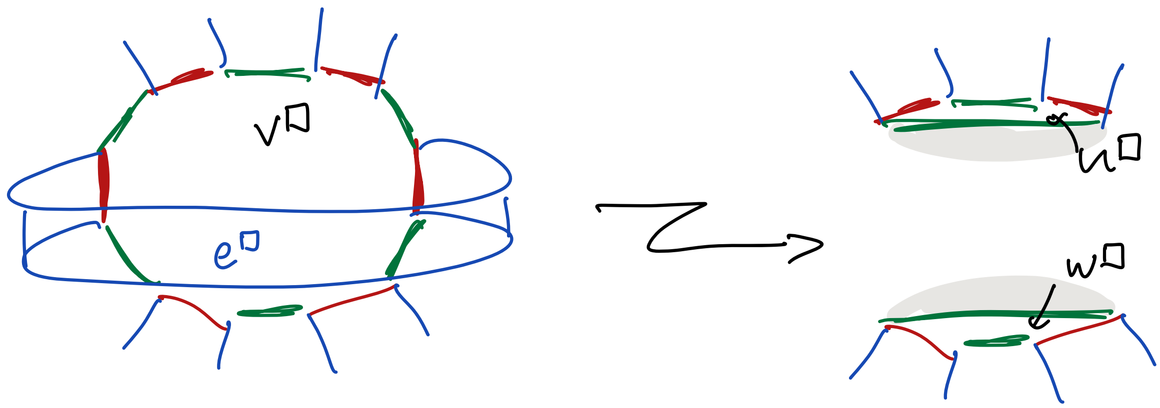


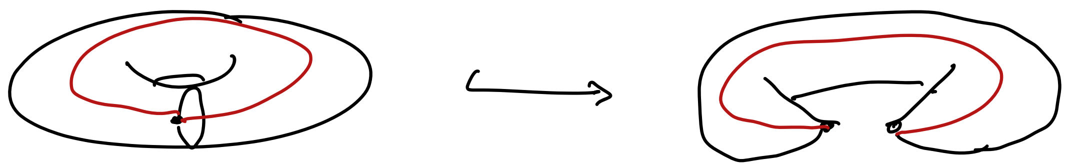
Algorithm for contraction:

$$(b/e)(\phi) = \begin{cases} b(a(b(a(b(\phi))))) & \text{if } b(\phi)\in e \text{ and } b(c(b(\phi)))\in e \\ b(a(b(\phi))) & \text{if } b(\phi)\in e \\ b(\phi) & \end{cases}$$









Contracting a 2-sided loop splits the vertex

V ++

e − −

F unchanged

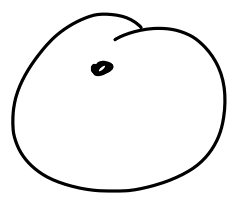
V−E+F increase by 2.

Contract any edge between two vertices

V−− e−− V−E+F unchanged

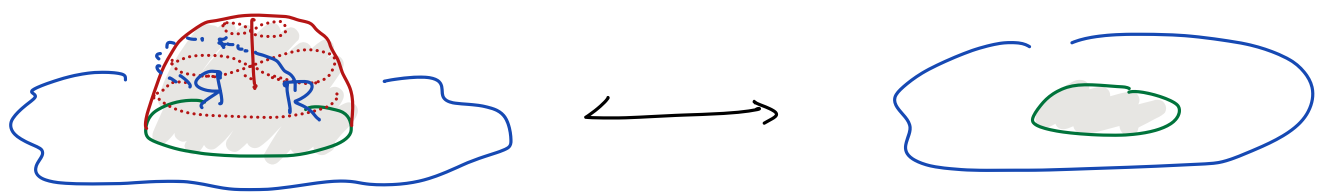
System of loops with 2 fewer edges.

Induction ⟶ base case

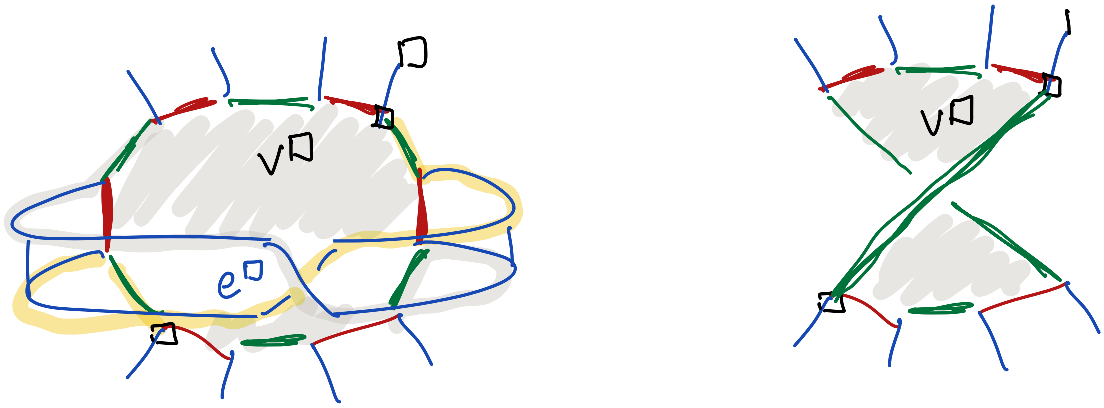


Lemma: Every orientable system of loops has even # loops $2g$

Surface is the (sphere + g handles) $\boxed{S(g,0)}$



twist
= Möbius band
= cross cap



Contracting a one-sided loop detaching a twist

$v=1$ $\quad e--$ $\quad f=1$ $\qquad V-E+F$ increased by 1

System of loops with one less loop.

Theorem Any system of loops can be reduced to trivial system

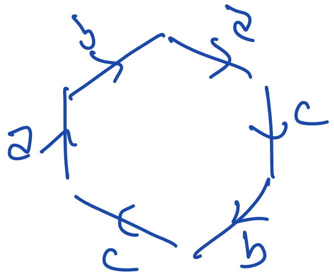
by contracting 1-sided then 2-sided

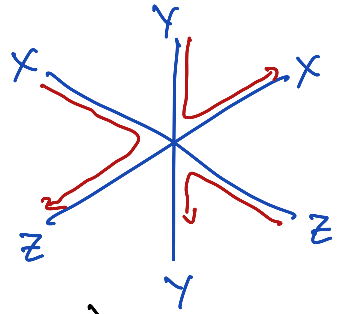
Any surface is sphere + g handles + h twists $\boxed{S(g,h)}$

$$\chi = 2-2g-h$$

Walter van Dyck

Dyck [({})[]{}](()(){})





$$S(1,1) = S(0,3)$$

$$\downarrow$$

$$S(g,h) = S(0, h+2g) \quad \text{if } h>0$$

Thm: Every surface is $S(g,0)$ for some $g \geq 0$
$S(0,g)$ for some $g > 0$

Euler's formula:

$$\chi(S(g,h)) = 2-2g-h$$

$$\chi(S) = \begin{cases} 2-2g & \text{if orientable} \\ 2-g & \text{if non-orientable} \end{cases}$$

Euler genus $\bar{g} = \begin{cases} 2g \\ g \end{cases}$ = # loop is system of loops

= #L in any tree-cotree decomp. (T, L, C).