CHAPTER 1

Simple Polygons

A book is man's best friend outside of a dog,
and inside of a dog it's too dark to read.

— Jim Brewer, *Boys Life* (February 1954)
Also attributed to Groucho Marx

QI thanks you for your question, and notes that some ebook readers do not require external illumination, e.g., the iPad. If one is swallowed Jonah-like by an enormous mammal one may continue to read unimpeded.

— Garson O'Toole, *Quote Investigator* (September 8, 2010)

The Jordan Curve Theorem and its generalizations are the formal foundations of many results, if not *every* result, in two-dimensional topology. In its simplest form, the theorem states that any simple closed curve partitions the plane into two connected subsets, exactly one of which is bounded. Although this statement is intuitively clear, perhaps even obvious, the generality of the phrase 'simple closed curve' makes a formal proof of the theorem incredibly challenging. A complete proof must work not only for sane curves like circles and polygons, but also for more exotic beasts like fractals and space-filling curves. Fortunately, these exotic curves rarely occur in practice, except as counterexamples in point-set topology textbooks.

Two examples illustrate the subtlety of this result. Many of Euclid's geometric proofs rely implicitly on the following assumption: If a line intersects one edge of a triangle but none of its vertices, that line must intersect one of the other two edges of the triangle. This assumption was first stated explicitly in 1882 by Pasch [28], who proved that it does not follow from Euclid's postulates; indeed, there are models of geometry that are consistent with Euclid's postulates but not with Pasch's axiom [33, 34].

path
endpoint
simple
To avoid excessive formality
closed curve
cycle
S1
simple
path-connected
component
polygonal chain
vertex!of a polygonal chain
edge!of a polygonal chain

A stronger version of the Jordan curve theorem states that the bounded component of the complement of a simple closed curve is *simply* connected, which means intuitively that it has no holes. As a second example of the subtlety of the Jordan curve theorem, consider the following plausible converse: *The boundary of every simply-connected planar region is a simple closed curve.* Readers are invited to think of their own counterexamples to this claim before turning to the end of the chapter.

A full proof of the Jordan Curve Theorem requires machinery that will appear only later in the book (specifically, homology). In this chapter we consider only one important special case: simple polygons. Polygons are by far the most common type of closed curve employed in practice, so this special case has immediate practical consequences. Most textbook proofs of the full Jordan Curve Theorem both dismiss this special case as trivial and rely on it as a key lemma. Indeed, the proof is ultimately elementary; nevertheless, its formal proof was the origin of several fundamental algorithmic tools in computational geometry and topology.[1]

1.1 A Few Definitions

We begin by carefully defining the terms of the theorem.

A ***path*** in the plane is an arbitrary continuous function $\pi\colon [0,1] \to \mathbb{R}^2$, where $[0,1]$ is the unit interval on the real line. The points $\pi(0)$ and $\pi(1)$ are called the ***endpoints*** of π; we say informally that π is a path from $\pi(0)$ to $\pi(1)$. A path π is ***simple*** if it is injective. To avoid excessive formality, we do not normally distinguish between a simple path (formally a function) and its image (a subset of the plane).

A ***closed curve*** (or ***cycle***) in the plane is a continuous function from the unit circle $\boldsymbol{S^1} := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ into X. A cycle is simple if it is injective. As for paths, we do not normally distinguish between a simple cycle and its image.

A subset X of the plane is ***(path-)connected*** if there is a path from any point in X to any other point in X. A ***(path-connected) component*** of X is a maximal path-connected subset of X.

The Jordan Curve Theorem. *The complement $\mathbb{R}^2 \setminus C$ of any simple closed curve C in the plane has exactly two components.*

A ***polygonal chain*** is a finite sequence of line segments $p_0p_1, p_1p_2, \ldots, p_{n-1}p_n$ joining adjacent pairs in a finite sequence of points $p_0, p_1, \ldots, p_n$ in the plane. The points p_i are the ***vertices*** of the polygonal chain, and the segments $p_{i-1}p_i$ are its ***edges***. Without loss of generality, we assume that no pair of consecutive edges is collinear; in particular, consecutive vertices are distinct.

Any polygonal chain describes a piecewise-linear path in the plane. Specifically, for any polygonal chain P of vertices $p_0, p_1, \ldots, p_n$, there is a unique path $\pi_P\colon [0,1] \to \mathbb{R}^2$ such that for any index i, the restriction of π_P to the interval $[i/n, (i+1)/n]$ is a linear map to the line segment $p_i p_{i+1}$:

$$\pi_p(t) = (nt \bmod 1)p_{\lceil nt \rceil} + ((1-nt) \bmod 1)p_{\lfloor nt \rfloor}$$

To avoid excessive formality, we rarely distinguish between a polygonal chain P (a sequence of line segments), the corresponding path π_P (a function into the plane).

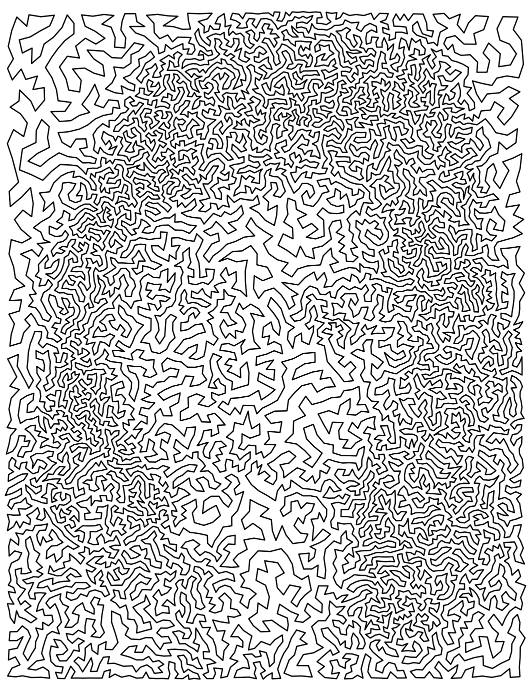
A polygonal chain is ***closed*** if it has at least one edge and its first and last vertices coincide (that is, if $p_0 = p_n$) and ***open*** otherwise. Closed polygonal chains are also called ***polygons***; a polygon with n vertices and n edges is also called an ***n*-gon**. Any closed polygonal chain describes (or less formally, ***is***) a closed curve in the plane.

A polygonal chain is ***simple*** if its vertices are distinct (except possibly $p_0 = p_n$) and its edges intersect only at endpoints, or equivalently, if the corresponding path or cycle is simple. To avoid excessive formality, we usually do not distinguish between a simple polygonal chain (formally a sequence of line segments) and the union of its edges (a subset of the plane). Every simple polygon has at least three edges.

To avoid excessive formality
polygonal chain!closed
polygonal chain!open
polygon
n-gon
simple
To avoid excessive formality

The Jordan Polygon Theorem. *The complement $\mathbb{R}^2 \setminus P$ of any simple polygon P in the plane has exactly two components.*

The phrase ‘simple polygon’ is more commonly used to describe the closed region bounded by a simple closed polygonal chain, but as we have not yet proved the Jordan polygon theorem, we don’t actually know that such a region always exists!



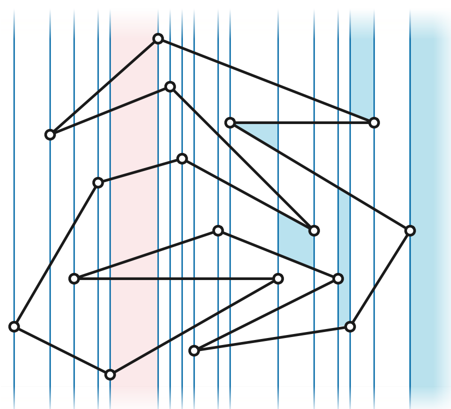
A simple 10000-gon.[2]

slab
trapezoid
floor
ceiling
wall

1.2 Proving the Jordan Polygon Theorem

Our proof of the Jordan polygon theorem very roughly follows an argument by Schönflies [12,29]. Fix an arbitrary simple polygon P with vertices $p_0, p_1, \ldots, p_{n-1}$. To simplify the proof, we assume without loss of generality that no two vertices p_i and p_j lie on the same vertical line; if we rotate P around any fixed point, there are only a finite number of orientations that violate this assumption, so we can always choose one that doesn't.

For each index i, let ℓ_i be the vertical line through vertex p_i. These n vertical lines subdivide the plane into $n+1$ ***slabs***, two of which are actually halfplanes. The edges of P further subdivide each slab into a finite number of ***trapezoids***. Some trapezoids are unbounded in one or more directions, and others actually triangles.



Vertical lines through the vertices of a simple polygon.
One slab and five trapezoids are shaded.

The boundary of each trapezoid consists of at most four line segments: the ***floor*** and ***ceiling***, which are segments of polygon edges, and the left and right ***walls***, which are (possibly infinite) segments of vertical lines ℓ_i whose endpoints (if any) lie on the polygon. Formally, we define each trapezoid to include its walls but not its floor, its ceiling, or any vertex on its walls. Thus, each trapezoid is connected, any two trapezoids intersect in a common wall or not at all, and the union of all the trapezoids is $\mathbb{R}^2 \setminus P$. Our proof assigns labels to each trapezoid in two different ways, which ultimately correspond to "inside P" and "outside P".

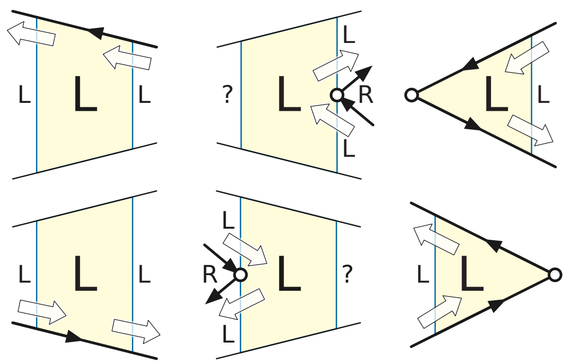
Lemma ≤ 2. *$\mathbb{R}^2 \setminus P$ has at most two components.*

Proof: Consider each edge $p_i p_{i+1}$ of the P to be directed from p_i to p_{i+1}. We label each trapezoid either *left* or *right*, depending on which side of the polygon the trapezoid appears. More formally, we label a trapezoid *left* if it satisfies at least one of the following conditions:

- The ceiling is directed from right to left.

- The floor is directed from left to right.
- The right wall contains a vertex p_i, and the incoming edge $p_{i-1}p_i$ is below the outgoing edge p_ip_{i+1}.
- The left wall contains a vertex p_i, and the incoming edge $p_{i-1}p_i$ is above the outgoing edge p_ip_{i+1}.

These conditions apply verbatim to unbounded trapezoids and to trapezoids that are actually triangles. There are four symmetric conditions for labeling a trapezoid *right*. Every trapezoid is labeled left or right *or possibly both*.



Left trapezoids.

Now we define a cyclic sequence of left trapezoids by traversing the polygon and listing the trapezoids we see to our left, as suggested by the arrows in the figure above. We start the traversal at vertex p_0 with the empty sequence of trapezoids. As we traverse each edge p_ip_{i+1} directed to the right (resp. to the left), we add all the trapezoids just above (resp. below) that edge to the sequence, in order from left to right (resp. right to left). When we traverse a vertex p_i whose neighbors p_{i-1} and p_{i+1} are both to the right of p_i, and the incoming edge $p_{i-1}p_i$ is above the outgoing edge p_ip_{i+1}, we add the trapezoid just to the left of p_i to the sequence. Similarly, when we traverse a right-extreme vertex where the polygon turns right, we add the trapezoid just to its right to the sequence. The resulting sequence of trapezoids contains every left trapezoid at least once (and at most four times); moreover, any adjacent pair of trapezoids in this sequence share a wall and thus have a connected union. It follows that the union of the left trapezoids is connected.

A symmetric argument implies that the union of the right trapezoids is also connected, which completes the proof. □

Lemma ≥ 2. *$\mathbb{R}^2 \setminus P$ has at least two components.*

Proof (Jordan): Label each trapezoid *even* or *odd* depending on the parity of the number of polygon edges directly above the trapezoid. Thus, within each slab, the highest trapezoid is even, and the trapezoids alternate between even and odd.

orientation!triple of points in the plane

Consider two trapezoids A and B that share a common wall, with A on the left and B on the right; suppose the wall lies on the vertical line ℓ_i. If the vertices p_{i-1} and p_{i+1} are on opposite sides of ℓ_i, exactly the same number of polygon edges are above A and above B. Suppose p_{i-1} and p_{i+1} lie to the left of ℓ_i. If the point p_i lies below the wall $A \cap B$, then A and B are below the same number of edges; otherwise, A is below two more edges than B. Similar cases arise when p_{i-1} and p_{i+1} lie to the right of ℓ_i. In all cases, A and B have the same parity.

It follows by induction that any two trapezoids in the same component of $\mathbb{R}^2 \setminus P$ have the same parity, which completes the proof. □

The Jordan Polygon Theorem follows immediately from Lemmas ≤ 2 and ≥ 2.

Our proofs of these lemmas apply more generally closed curves that have a finite number of left- and right-extrema and a finite number of intersections with any vertical line—for example, any closed curve composed of a finite number of polynomial paths. However, considerably more work is required to handle *arbitrary* closed curves.[3]

The two lemmas actually rely on complementary properties of closed curves in the plane. Lemma ≤ 2 is still true if we replace the plane with any other topological surface. On the other hand, our proof of Lemma ≥ 2 applies nearly verbatim if we replace P with any set of (not necessarily simple) polygons; the only change is that we must also draw vertical lines through intersection points.

1.3 Point in Polygon Test

It is straightforward to convert the proof of Lemma ≥ 2 into the standard algorithm to test whether a point is inside a simple polygon in the plane in linear time: Shoot an arbitrary ray from the query point, count the number of times this ray crosses the polygon, and return True if and only if this number is odd. This algorithm has been rediscovered several times, but the earliest published description seems to be a 1962 paper of Shimrat [31] (later corrected by Hacker [13]).

To make the ray-parity algorithm concrete, we need one numerical primitive from computational geometry. A triple (q, r, s) of points in the plane is ***oriented counterclockwise*** if walking from q to r and then to s requires a left turn, or oriented clockwise if the walk requires a right turn. More formally, consider the 3×3 determinant

$$\begin{aligned}\Delta(q,r,s) := \det \begin{bmatrix} 1 & q.x & q.y \\ 1 & r.x & r.y \\ 1 & s.x & s.y \end{bmatrix} \\ = (r.x - q.x)(s.y - q.y) - (r.y - q.y)(s.x - q.x).\end{aligned}$$

The triple (q, r, s) is oriented counterclockwise if $\Delta(q, r, s) > 0$ and oriented clockwise if $\Delta(q, r, s) < 0$. If $\Delta(q, r, s) = 0$, the three points are collinear. The orientation of a triple of points in unchanged by any cyclic permutation, but reversed by swapping any two points.

Finally, here is the algorithm. The input polygon P is represented by an array of consecutive vertices, which are assumed to be distinct. The algorithm returns $+1$, -1, or 0 to indicate that the query point q lies inside, outside, or directly on P, respectively. The function call OnOrBelow(q, r, s) returns -1 if q lies directly below the segment rs, returns 0 if q lies on rs, and returns $+1$ otherwise. To correctly handle degenerate cases, OnOrBelow treats any polygon vertex on the vertical line through q as though it were slightly to the left. The algorithm clearly runs in $O(n)$ time.

```
PointInPolygon(P[0..n−1], q):
    sign ← −1
    P[n] ← P[0]
    for i ← 0 to n − 1
        sign ← sign · OnOrBelow(q, P[i], P[i+1])
    return sign
```

```
OnOrBelow(q, r, s):
    if r.x < s.x
        swap r ⟷ s
    if (q.x < s.x) or (q.x ≥ r.x)
        return +1
    return sgn(Δ(q, r, s))
```

1.4 Triangulations

triangulation!in the plane
vertex!of a polygon triangulation
edge!of a polygon triangulation
face!of a polygon triangulation
triangulation!of a polygon
exercises for the reader

Most algorithms that operate on simple polygons do not only store the sequence of vertices and edges, but also a decomposition of the polygon into simple pieces that are easier to manage. In the most natural decomposition, the pieces are triangles that meet edge-to-edge. More formally, a ***triangulation*** is a triple of sets (V, E, T) with the following properties.

- V is a finite set of points in the plane, called ***vertices***.
- E is a set of interior-disjoint line segments, called ***edges***, between points in V.
- T is a set of interior-disjoint triangles, called ***faces***, whose vertices are in V and whose edges are in E.
- Every point in V is a vertex of at least one triangle in T.
- Every segment in E is an edge of at least one triangle in T.

If in addition the union of the triangles in T is the closure of the interior of a simple polygon P, we call (V, E, T) a ***triangulation of P***. More generally, we call any triangulation (V, E, T) a triangulation of the set $\bigcup T$. These definitions have several immediate consequences, whose proofs we leave as exercises for the reader.

Lemma 1.3. *Let $\Delta = (V, E, T)$ be any triangulation of any simple polygon P.*

(a) Every vertex of P is also a vertex of Δ.

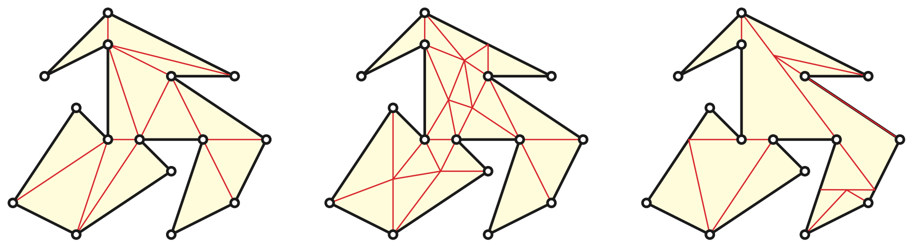
(b) Every edge of P is the union of vertices and edges of Δ.

(c) Every segment in E is an edge of either one triangle or two triangles in T.

(d) The intersection of any two triangles in T is either an edge of both triangles, a vertex of both triangles, or the empty set.

frugal triangulation
diagonal
ear

We call a polygon triangulation ***frugal*** if the vertices of the triangulation are precisely the vertices of P. A ***diagonal*** of a simple polygon P is a line segment whose endpoints are vertices of P and otherwise lies in the interior of P. Every edge of a frugal triangulation is either an edge of the polygon or a diagonal. We emphasize that our definitions of triangulations and diagonals require the Jordan polygon theorem.



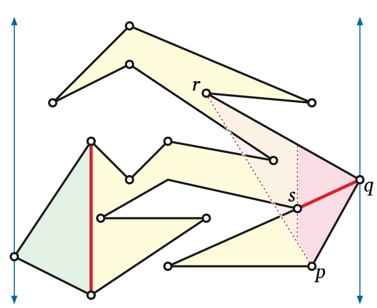
From left to right: A frugal triangulation, a non-frugal triangulation, and a non-triangulation.

After playing with a few examples, it may seem obvious that every simple polygon has a frugal triangulation, but a formal proof of this fact is surprisingly subtle.[4]

Lemma 1.4 (Dehn [9], Lennes [23]). *Every simple polygon with at least four vertices has a diagonal.*

Proof: Let q be the rightmost vertex of P, and let p and r be the vertices immediately before and after q in order around P.

First suppose the segment pr does not otherwise intersect P. For any point x in the interior of pr, the ray from x through q crosses P exactly once, at the point q. (The Jordan *triangle* theorem implies that P does not intersect the interior of the the segment xq, or more generally the interior of the triangle $\triangle pqr$. Similarly, the ray from q leading directly away from x does not intersect P, because q is the leftmost vertex of P.) It follows that pr lies in the interior of P and thus is a diagonal. In this case, we call $\triangle pqr$ an ***ear*** of P.



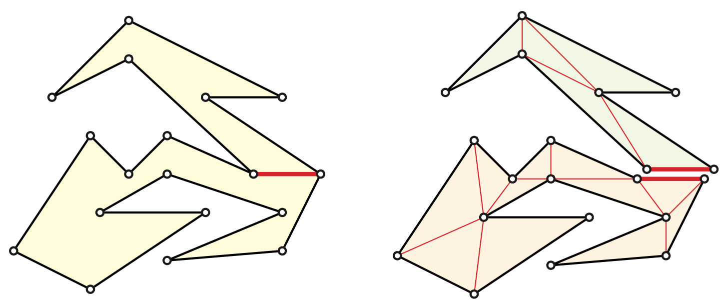
The leftmost vertex is the tip of an ear. The rightmost vertex is not.

Otherwise, P intersects the interior of pr. In this case, the Jordan *triangle* theorem implies that $\triangle pqr$ contains at least one vertex of P in its interior. Let s the rightmost vertex in the interior of $\triangle pqr$. (In fact, we can take s to be any vertex in the interior of $\triangle pqr$ such that some line through s separates q from all other vertices in the interior of $\triangle pqr$.) The line segment qs lies in the interior of P and thus is a diagonal. □

One can verify mechanically

Theorem 1.5 (Dehn [9], Lennes [23]). *Every simple polygon with n vertices has a frugal triangulation.*

Proof: The result follows by induction from the previous lemma. Let P be a simple polygon with n vertices $p_0, p_1, p_2, \ldots, p_{n-1}$. If $n = 3$, then P is a triangle, and thus has a trivial triangulation. Otherwise, suppose without loss of generality (reindexing the vertices if necessary) that $p_0 p_i$ is a diagonal of P, for some index i. Let P^+ and P^- denote the polygons with vertices $p_0, p_i, p_{i+1}, \ldots, p_{n-1}$ and $p_0, p_1, p_2, \ldots, p_i$, respectively. The definition of diagonal implies that both P^+ and P^- are simple.



Recursively triangulating a polygon.

Consider an arbitrary point p in the interior of P but not on the diagonal $p_0 p_i$, and an arbitrary ray R from p that does not intersect $p_1 p_i$. The definition of 'interior' implies that R crosses the boundary of P an odd number of times. Thus, either R crosses the boundary of P^+ an odd number of times and crosses the boundary of P^- an even number of times, or vice versa. We conclude that every point in the interior of P is either in the interior of P^+, in the interior of P^-, or on the diagonal $p_0 p_i$.

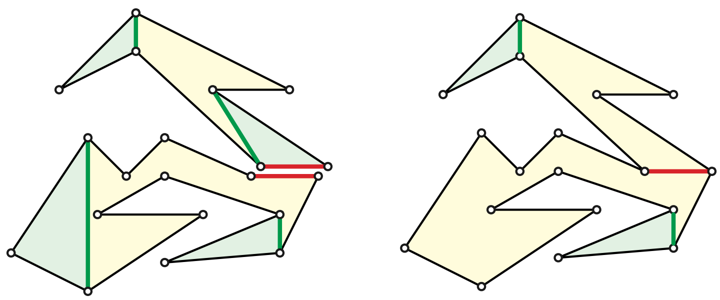
The inductive hypothesis implies that P^+ has a frugal triangulation (V^+, E^+, T^+) and that P^- has a frugal triangulation (V^-, E^-, T^-). One can verify mechanically that $(V^+ \cup V^-, E^+ \cup E^-, T^+ \cup T^-)$ is a frugal triangulation of P. □

Corollary 1.6 (Dehn [9], Meisters [25]). *Every simple polygon with at least four vertices has an ear.*

Proof: We prove the following stronger claim by induction: Every simple polygon with at least four vertices has two ears that do not share an edge. Fix a simple polygon P with at least four vertices, and let pq be a diagonal of P. Following the proof of Theorem 1.5, splitting P along pq yields two simple polygons P^+ and P^-. If P^- is a triangle, its two edges other than pq define an ear of P. Otherwise, the inductive hypothesis implies that P^- has two ears that do not share an edge. At most one of these ears uses the

exercises for the reader
trapezoidal decomposition

edge pq, so at least one is also an ear of P. In either case, we conclude that P^- contains an ear of P. A symmetric argument implies that P^+ contains an ear of P. These two ears have no edge in common. □



Recursively finding finding two disjoint ears.

We leave the following additional observations as exercises for the reader, hint, hint.

Lemma 1.7. *Every frugal triangulation of a simple n-gon has exactly $n-2$ facets and exactly $n-3$ edges.*

Lemma 1.8. *Let P be a simple polygon with vertices $p_0, p_1, \dots, p_{n-1}$. Let i, j, k, l be four distinct indices with $i < j$ and $k < l$, such that both p_ip_j and p_kp_l are diagonals of P. These two diagonals cross if and only if either $i < k < j < l$ or $k < i < l < j$.*

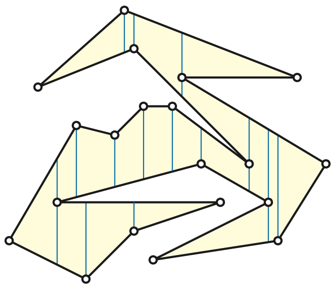
Lemma 1.9. *Any maximal set of non-crossing diagonals in a simple polygon P is the edge-set of a frugal triangulation of P.*

1.5 Constructing a Triangulation

The proof of Lemma 1.4 implies an algorithm to find a diagonal in a simple polygons with n vertices in $O(n)$ time. By applying this algorithm recursively, we can compute a triangulation of any simple n-gon in $O(n^2)$ time. We can dramatically improve this algorithm by taking a more global view; consider the following alternative proof of Lemma 1.4.

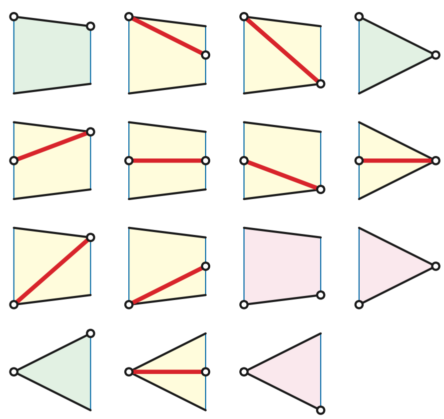
Fix a simple polygon P with vertices $p_0, p_1, \dots, p_{n-1}$ for some $n \geq 4$; as usual, we assume without loss of generality that no two vertices of P lie on a common vertical line. We begin by subdividing the closed interior of P into trapezoids with vertical line *segments* through the vertices. Specifically, for each vertex p_i, we cut along the longest vertical segment through p_i in the closed interior of P. The resulting subdivision, which is called a ***trapezoidal decomposition*** of P, can also be obtained from the slab decomposition we used to prove the Jordan polygon theorem, by removing every exterior wall and every wall that does not end at a vertex of P.

monotone mountain
convex
exercise for the reader



A trapezoidal decomposition of a simple polygon.

We can identify fifteen different types of trapezoids, depending on whether the left and right walls have a vertex at the ceiling, have a vertex at the floor, have a vertex in the interior, or consist entirely of a vertex. (There are fifteen types instead of sixteen because at least one of the walls of a trapezoid is not a single point.) In nine of these fifteen cases, the line segment between the two vertices on the boundary of the trapezoid lies entirely in the interior of the trapezoid and thus is a diagonal of P.

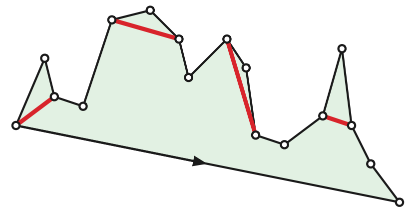


Fifteen types of trapezoids, nine of which contain diagonals

If none of those nine types of trapezoids occur in the decomposition, then either every trapezoid has both vertices on the ceiling, or every vertex has both vertices on the floor. Then P is a special type of polygon called a ***monotone mountain***: any vertical line crosses at most two edges of P, and the leftmost and rightmost vertices of P are connected by a single edge.

Without loss of generality, suppose p_0 is the leftmost vertex, p_{n-1} is the rightmost vertex, and every other vertex is above the edge p_0p_{n-1} (so all trapezoids have both vertices on the ceiling). Call a vertex p_i ***convex*** if the interior angle at that vertex is less than π, or equivalently, if the triple (p_{i-1}, p_i, p_{i+1}) is oriented *clockwise*. Let p_i be any convex vertex other than p_0 and p_1; for example, take the vertex furthest above the line $\overleftrightarrow{p_0p_1}$. Then the line segment $p_{i-1}p_{i+1}$ is a diagonal; we leave the proof of this final claim as an exercise for the reader.

homeomorphism
homeomorphic



A monotone mountain and four of its diagonals.

This completes our second proof of Lemma 1.4.

Construct trapezoidal decomposition in $O(n \log n)$ time by plane sweep. Also cite randomized incremental construction. Sketch only; a full description requires data structures introduced in the next chapter. See Chapter 3 for existing sketch.

Given a trapezoidal decomposition of P, we can construct a frugal triangulation of P in $O(n)$ time using the proof of Lemma 1.4 as follows. First, we identify all trapezoids in the decomposition that contain a diagonal. These diagonals partition the polygon into a set of monotone mountains, some or all of which may be triangles. Then we triangulate each monotone mountain by repeatedly cutting off convex vertices, using the following algorithm.

Three-penny algorithm. Leave as exercise?

1.6 The Jordan-Schönflies Theorem

Now we consider the following very useful extension of the Jordan curve theorem, attributed to Arthur Schönflies [30]. A ***homeomorphism*** is a continuous function whose inverse is also continuous. Two spaces are ***homeomorphic*** if there is a homeomorphism from one to the other. For example, a simple closed curve can be defined as any subset of the plane homeomorphic to the circle S^1.

The Jordan-Schönflies Theorem. *For any simple closed curve C in the plane, there is a homeomorphism from the plane to itself that maps C to the unit circle S^1.*

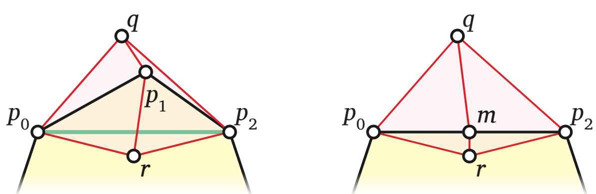
This theorem implies not only that $\mathbb{R}^2 \setminus C$ has two components, but also that C is the boundary of both components, and that the closure of the bounded component is homeomorphic to the unit disk $B^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Again, these results seem obvious at first glance, but the natural three-dimensional generalization of the Jordan-Schönflies theorem is actually false.

Like the Jordan curve theorem, a full proof of the Jordan-Schönflies theorem requires more advanced tools; we will prove only the special case where C is a simple polygon. In fact, this is the only case that Schönflies [30] actually proved, and a proof of this special case already appears in Dehn's unpublished manuscript [9]. The proofs presented in this chapter (including the triangulation results in the previous section) are modeled after Dehn's arguments, but with several further simplifications.

The Dehn-Schönflies Polygon Theorem. *For any simple polygon P in the plane, there is a homeomorphism from the plane to itself that maps P to the boundary of a triangle.*

Proof (Dehn [9]): We prove the theorem by induction. Fix a simple polygon P with vertices $p_0, p_1, \ldots, p_{n-1}$. If P is a triangle, the theorem is trivial, so assume that $n \geq 4$.

Without loss of generality, suppose p_1 is the tip of an ear of P. Let P' denote the polygon with vertices $p_0, p_2, p_3, \ldots, p_{n-1}$; the definition of 'ear' implies that P' is simple. The induction hypothesis implies that there is a homeomorphism $\phi' \colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps P' to the boundary of a triangle. Thus, it suffices to prove that there is a a homeomorphism $\psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps P to P'; the composition $\phi = \phi' \circ \psi$ is a homeomorphism of the plane that maps P to the boundary of a triangle.



Shrinking an ear.

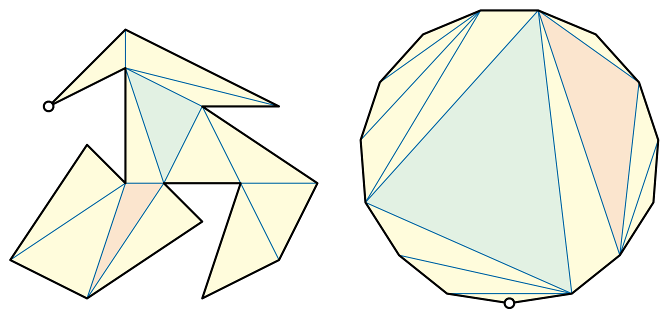
We construct a suitable piecewise-affine homeomorphism ψ as illustrated in the figure above. Let q be a point close to p_1 but outside P; let m be the midpoint of the diagonal p_0p_2; and let r be a point close to m, just outside the triangle $\triangle p_0p_1p_2$. Finally, let Q denote the closed convex quadrilateral with vertices p_0, q, p_2, r. We have two different but combinatorially isomorphic triangulations of Q, one with internal vertex p_1, the other with internal vertex m. Let ψ to be the unique piecewise-affine map that maps the first triangulation of Q to the second. That is, we set $\psi(x)$ to the identity outside the interior of Q; we set $\psi(p_1) = m$; and then we linearly extend ψ across the triangles $\triangle p_0p_1q$, $\triangle qp_1p_2$, $\triangle p_2p_1r$, and $\triangle rp_1p_0$. It is routine to verify that ψ is indeed a homeomorphism. □

For many applications of this theorem, the following weaker version is actually sufficient.

Theorem 1.10. *The closure of the interior of any simple polygon is homeomorphic to a triangle.*

Proof: Let P be a simple polygon with n vertices $p_1, p_2, \ldots, p_n$, and let Q be a *convex* n-gon with vertices $q_1, q_2, \ldots, q_n$. We first proving that the closed interior of P is homeomorphic to the closed interior of Q, and then prove that the the closed interior of Q is homeomorphic to a triangle.

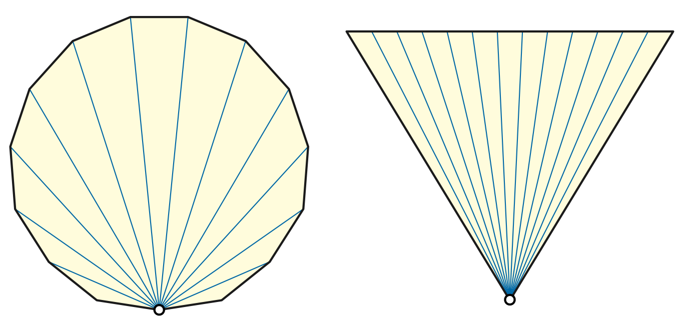
Let (V, E, T) be any frugal triangulation of P, and let (V', E', T') be the triple obtained from (V, E, T) by replacing each point p_i with the corresponding point q_i. That is, $q_i q_j \in E'$ if and only if $p_i p_j \in E$, and $\triangle q_i q_j q_k \in T'$ if and only if $\triangle p_i p_j p_k \in T$. Convexity implies that any line segment between non-adjacent vertices of Q is a diagonal of Q. Moreover, Lemma 1.8 implies that for any pair $p_i p_j$ and $p_k p_l$ of noncrossing diagonals of P, the corresponding diagonals $q_i q_j$ and $q_k q_l$ of Q are also noncrossing. It follows that (V', E', T') is a frugal triangulation of Q.



Any triangulation of a simple polygon is isomorphic to a triangulation of a regular polygon.

Now let ϕ be the piecewise-linear map from the closed interior of P to the closed interior of Q that maps each vertex p_i onto the corresponding vertex q_j, and moreover maps each triangle $p_i p_j p_k$ in T *affinely* onto the corresponding triangle $q_i q_j q_k$ in T'. It is straightforward to check that ϕ is a homeomorphism.

Finally, consider a new triangulation of Q obtained by adding diagonals from q_n to every other vertex except q_{n-1} and q_1. As in the previous paragraph, we can easily construct a piecewise-linear homeomorphism from this triangulation to a (non-frugal) triangulation of any triangle, with one edge of the triangle subdivided into $n-2$ smaller collinear segments. □



Transforming a regular polygon into a triangle.

1.7 Generalizations of Polygons

simple polygon
boundary X
interior X
polygon with holes
outer boundary!of polygon with holes
inner boundary!of polygon with holes
planar straight-line graph
vertex
edges
face!of a planar graph

From now on, following standard usage, we define a ***simple polygon*** to be the closure of the interior of a simple closed polygonal chain. For any simple polygon P, we write $\boldsymbol{\partial P}$ to denote its boundary (a simple closed polygonal chain) and $\boldsymbol{P^\circ}$ to denote its interior (homeomorphic to an open disk).

A ***polygon with holes*** is a set of the form $P = P_0 \setminus (P_1^\circ \cup P_2^\circ \cup \cdots \cup P_h^\circ)$, where P_0 is a simple polygon and $P_1, \ldots, P_h$ are disjoint simple polygons, called *holes*, in the interior of P_0. Again, if P is a polygon with holes, we write ∂P and P° to denote the boundary and interior of P, respectively. Observe that $\partial P = \partial P_0 \cup \partial P_1 \cup \cdot \cup \partial P_h$. We sometimes call ∂P_0 the ***outer boundary*** and each ∂P_i an ***inner boundary*** of P.

Not surprisingly, polygons with holes also admit frugal triangulations. Adapting our initial inductive proof of Theorem 1.5 to this setting requires some subtle modifications, but our second proof using trapezoidal decompositions applies with no changes at all.

In fact, our second proof directly implies an even more general result. A ***planar straight-line graph*** consists of a set V of points in the plane, called ***vertices***, and a set E of interior-disjoint line segments with endpoints in V, called ***edges***. A ***face*** of a planar straight-line graph G is any component of its complement $\mathbb{R}^2 \setminus G$. In a *triangulation* of a face f, the union of the triangles is the closure of f, every segment in E that lies on the boundary of f is covered by edges of the triangulation, and every vertex of f is a vertex of some triangle. As usual, a triangulation is *frugal* if every vertex of the the triangulation lies in V. Our earlier arguments imply the following result:

Theorem 1.11. *Every bounded face of every planar straight-line graph has a frugal triangulation. Moreover, given a planar straight-line graph G consisting of at most n vertices and edges, we can construct a frugal triangulation of* ***every*** *bounded face of G in $O(n \log n)$ time.*

Exercises

1. Collected "exercises for the reader":
   a) Prove Lemma 1.3.
   b) Prove Lemma 1.7.
   c) Prove Lemma 1.8.
   d) Prove Lemma 1.9.
   e) Prove that any convex vertex of a monotone mountain, except possibly the base vertices, is the tip of an ear.
2. Sketch an algorithm to triangulate any given monotone mountain, specified by a sequence of n vertices, in $O(n)$ time.
3. Sketch an algorithm to determine in $O(n \log n)$ time whether a given polygon, specified by a sequence of n vertices, is simple.

4. Our proof of the Dehn-Schönflies theorem produces, for any simple polygon P, a *piecewise-linear* or *PL* homeomorphism ϕ from the plane to itself that maps P to a triangle. That is, there is a triangulation Δ of the plane (or more formally, of a very large rectangle) such that the restriction of ϕ to any triangle in Δ is affine. The *complexity* of ϕ is the minimum number of triangles in such a triangulation.
   a) Prove that the composition of two PL homeomorphisms of the plane is another PL homeomorphism.
   b) Suppose ϕ is a PL homeomorphism with complexity x, and ψ is a PL homeomorphism with complexity y. What can you say about the complexity of the PL homeomorphism $\phi \circ \psi$?
   c) Prove that for any simple n-gon P, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n)$ that maps the polygon P to a triangle.
   d) Prove that for any two simple n-gons P and Q, there is a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity $O(n^2)$ such that $\phi(P) = Q$. *[Hint: Use the previous part.]*
   e) Prove that the $O(n^2)$ bound in the previous problem is tight. That is, for any integer n, describe two simple n-gons P and Q, such that any piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with $\phi(P) = Q$ has complexity $\Omega(n^2)$.
   f) **Open problem:** Prove that finding the minimum-complexity homeomorphism of the plane that maps one given simple polygon to another is NP-hard.

Notes

1. (page 2) The subtlety of the Jordan Curve Theorem leads to considerable confusion about its first correct proof, in no small part because the first proofs appeared when the formal axiomatic foundations of analysis and topology were still being developed.

Although it was used implicitly by several earlier mathematicians, the Jordan curve theorem was first formally stated in the early 1800s by Bolzano [5, 6], who recognized that no proof was known at the time. The first proof of the theorem was given by Jordan about 50 years later, in a set of lecture notes on analysis [19, 20]. Jordan's proof was sketchy, especially by the exacting standards of early 20th century formalists; in particular, he asserted without proof that the theorem is true for simple polygons. Although many contemporaries of Jordan agreed that his reduction from general curves to polygons was correct [14, 27], most modern sources simultaneously dismiss the polygonal case as trivial and starkly report that Jordan's proof was "invalid" [8], "faulty" [32], or even "completely wrong" [10]. More than a century later, Hales [16] argued convincingly that Jordan's proof, while sketchy, was essentially correct. In particular, his proof does not actually rely on his unproved assertion that the theorem holds for simple polygons, but rather on a weaker claim (called Lemma ≥ 2 here), which he does prove.

The first attempt at a proof of the Jordan curve theorem was by Schönflies [29]. His proof is essentially correct for polygons and other well-behaved curves, but as later pointed out by Brouwer [7], the general proof has several fundamental flaws. Brouwer's paper [7] ends with the following rather devastating summary:

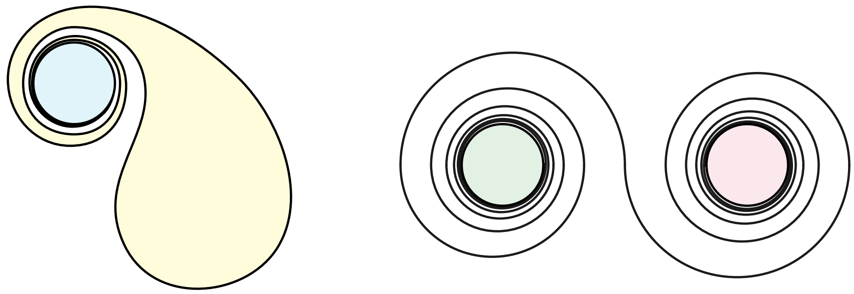
> Among Schönflies's theorems (either derived or as obviously applied) the following are incorrect:
>
> - that any closed curve can be decomposed into two proper arcs;
> - that in the decomposition of a closed curve into two arcs, at least one of them is proper;
> - that each compact, simply connected region has a closed outer boundary curve;
> - that each compact, simply connected region is uniquely determined by its boundary;
> - that each compact, closed set that is the common boundary of two areas without common points is a closed curve;
>
> While the following remain uncertain:
>
> - that the one-to-one continuous image of a closed curve is also a closed curve;
> - that the one-to-one continuous image of a curved surface is also a curved surface.

Veblen softened his blow somewhat with a conciliatory footnote on the first page:

> I point out explicitly that this work is not meant to undercut the high value of Schönflies' investigations. Only their broad scope has led me to this criticism, which incidentally does not apply significantly to the largest part, namely the theory of simple curves.

Schönflies' proof relied on metrical properties of the plane. The first axiomatic proof of the Jordan *polygon* theorem appears in an unpublished manuscript of Dehn [9], almost certainly written in 1899, while he was still a graduate student [12]; in fact, this manuscript includes a proof of the stronger "Jordan-Schönflies theorem" for polygons, which we describe in Section 1.6. Axiomatic proofs for simple polygons and other well-behaved curves were published roughly simultaneously by Lennes (in his master's thesis) [21, 23], Veblen (in his PhD thesis) [38], Ames (in his PhD thesis) [2, 3], Bliss [4], and Hahn [14]. (Nels Johann Lennes and Oswald Veblen were both PhD students of Eliakim H. Moore at the University of Chicago in the early 1900s. According to Veblen [39], the Jordan polygon theorem was discussed at Moore's 1901–02 seminar on the foundations of geometry. most likely with both Lennes and Veblen in attendance.)

Most modern sources (at least prior to Hales) give Oswald Veblen credit for the first rigorous axiomatic proof of the complete Jordan curve theorem [39]; however, Veblen's proof was not without flaws. Both Hahn [14] and Lennes [23] found and repaired gaps in Veblen's proof; in particular, Lennes observed that the axiomatic system underlying Veblen's argument assumed without proof that every simple polygon can be triangulated. Hahn's proof of the Jordan polygon theorem carefully bypassed the need



Open curves that split the plane into three components, after Brouwer [7].
The yellow region on the left is simply-connected, but its boundary is not a simple closed curve.

for triangulation [14]. Lennes gave the first published proof of the triangulation theorem [21, 23], followed quickly by his own proof of the full Jordan curve theorem [22]. However, a proof of the triangulation theorem already appears in Dehn's unpublished manuscript [9, 12]. (The polygon triangulation theorem has its own tortured history, dating back to unpublished notes of Gauss [11], and with new incorrect proofs appearing in the literature well into the 20th century [17].)

Since this early work, dozens of different proofs of the Jordan curve theorem have appeared in the literature; see, for example, the short proof by Tverberg [37], a reduction to Brouwer's fixed point theorem by Maehara [24], a proof of a much more general theorem by Thomassen [35], and a formal (computer-verified) proof by Hales [15].

2. (page 3) This figure was created using Windell Oskay's open-source application StippleGen (http://www.evilmadscientist.com/2012/stipplegen2/) from an excerpt of a pastel drawing by Connie Erickson; the image is used here with the artist's permission.

3. (page 6) As a particularly bad example of a closed curve for which the proofs of Lemmas ≤ 2 and ≥ 2 both fail, consider Osgood's family of curves with positive area (Lebesgue measure) [26].

4. (page 8) Hu [18] and Toussaint [36] survey several incorrect published proofs of the polygon triangulation theorem.

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