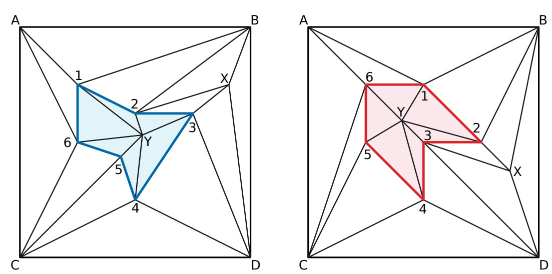
1. For any simple polygons P and Q, the Dehn-Schönflies theorem implies that there is a homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(P) = Q$. Moreover, if P has n vertices $p_1, p_2, \ldots, p_n$ and Q has n vertices $q_1, q_2, \ldots, q_n$, we can further require that $\phi(p_i) = q_i$ for every index i. This question asks you to construct such a homeomorphism explicitly.

   Let $\square$ be a square that is large enough to comfortably contain both P and Q. We say that a triangulation T of $\square$ ***supports*** P if every vertex of P is a vertex of T and every edge of P is the union of edges of T—more succinctly, if some subcomplex of T is a triangulation of P. Two triangulations T_P and T_Q of $\square$ with labeled vertices are ***compatible*** with P and Q if they satisfy the following conditions:

   - T_P and T_Q are isomorphic as labeled planar maps. That is, the vertex labeling induces bijections between the vertices, edges, and faces of T_P and the vertices, edges, and faces of T_Q, respectively.
   - Corresponding vertices on the boundary of $\square$ have the same coordinates in both triangulations.
   - T_P supports P and T_Q supports Q.
   - The vertex labeling also induces bijections between the vertices, edges, and interior faces of P and the vertices, edges, and interior faces of Q, respectively. In particular, for any index i, vertices p_i and q_i have the same label in T_P and T_Q, respectively.



Compatible labeled triangulations of two simple polygons.

   (a) Describe an algorithm to compute compatible triangulations for two given n-gons with at most $O(n^2)$ vertices. (This implies a piecewise-linear homeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with complexity at most $O(n^2)$ that is the identity outside the bounding box $\square$.)

   (b) Prove that the $O(n^2)$ upper bound cannot be improved in the worst case.

   ⋆(c) Describe and analyze an algorithm to determine if two simple n-gons have compatible triangulations with exactly $n + 4$ vertices: the vertices of the polygon plus the vertices of the bounding box $\square$.[1]

   ★(d) Prove that computing compatible triangulations with the minimum number of vertices is NP-hard.[2]

   It may be easier to start by considering compatible triangulations only of the interiors of the polygons. See Aronov, Seidel, and Souvaine [*CGTA* 1993]. A similar problem for arbitrary point sets was previously considered by Saalfeld [SOCG 1989].

[1]Small stars indicate problems I don't know how to solve. That does not necessarily mean the problem is open, difficult, or interesting.

[2]Large stars indicate problems that I know are open.

2. Recall that the dual of a directed edge $e = \mathit{tail}(e) \to \mathit{head}(e)$ in a directed plane graph is $e^* = \mathit{left}(e)^* \to \mathit{right}(e)^*$.

   (a) Prove that a directed plane graph G is acyclic if and only if the dual graph G^* is strongly connected.

   (b) Call an edge of a directed graph G *internal* if its endpoints lie in the same strong component of G and *external* otherwise. Prove that an edge e in a directed *plane* graph G is internal if and only if the corresponding dual edge e^* of the dual graph G^* is external.

3. A vertex v of a directed plane graph is *regular* if all incoming edges are adjacent in cyclic order around v; a non-regular vertex is called a *saddle*.

   (a) Let G be a planar dag with a unique source and a unique sink. Prove that any planar embedding of G has no saddle vertices.

   (b) Let G be a planar dag with s sources and t sinks. Prove that any planar embedding of G has at most $s + t - 2$ saddles.