These questions are rather long, so I don't expect everyone to submit solution to everything.

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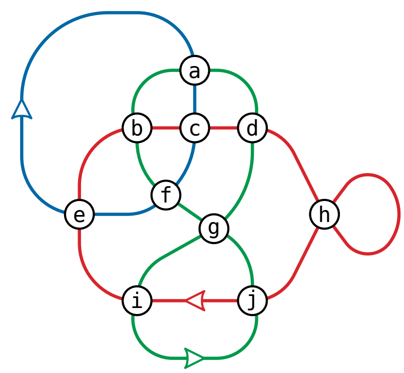
1. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be a set of generic closed curves that intersect each other only pairwise, transversely, and away from their vertices; we call any such set a *(generic) family of planar curves*. For simplicity, assume that every curve γ_i is either non-simple or intersects another curve γ_j. Then the image of Γ is a 4-regular plane graph, which may be disconnected. Conversely, every 4-regular plane graph is the image of a family of planar curves.

   A *Gauss paragraph* of Γ is a family of k strings, obtained by uniquely labeling the vertices of Γ, and then listing the vertices in order along each curve γ_i in an arbitrary direction, starting at an arbitrary basepoint, considering the curves in arbitrary order.

   A connected family of planar curves with Gauss paragraph {`acfe`, `jgfbadgi`, `iebcdhhj`}.

   Now let's go the other direction. A Gauss paragraph is any set of non-empty strings in which any symbol appears exactly twice or not at all. Each Gauss paragraph X defines a 4-regular graph $G(X)$ whose vertices are the characters in X and whose edges correspond to adjacent character pairs. If X is the Gauss paragraph of a family of planar curves, then $G(X)$ is the image graph of that family.

   (a) Prove that if X is the Gauss paragraph of a family of planar curves, then the edges of $G(X)$ can be directed so that every vertex has in-degree 2 and out-degree 2. *[Hint: Consider self-intersection points of one curve and intersection points of two curves separately.]*

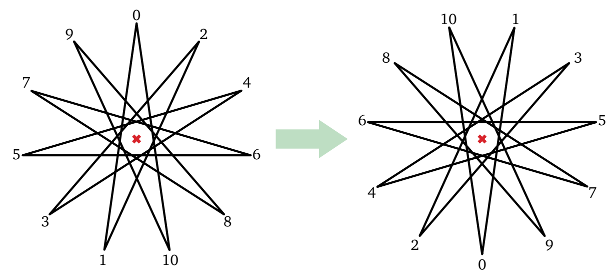
   (b) Describe a linear-time algorithm that either directs the edges of $G(X)$ as described in part (a) or correctly reports that no such orientation exists, given the Gauss paragraph X as input.

   (c) Prove that X is the Gauss paragraph of a generic family of planar curves if and only if $G(X)$ has a planar embedding such that every component has a weakly simple Euler tour.

   (d) Sketch a linear-time algorithm to decide whether a given string is the Gauss paragraph of a generic family of planar curves. (Just describe the necessary modifications to the algorithm for single curves.)

2. Fix an arbitrary point o in the plane, called the *origin*. Let P be a polygon in $\mathbb{R}^2 \setminus o$ with vertices $p_0, p_1, \ldots, p_{n-1}$. A *vertex move* on P replaces an arbitrary vertex p_i with a new point q_i. This vertex move is *safe* if neither of the triangles $\triangle p_i q_i p_{i-1}$ and $\triangle p_i q_i p_{i+1}$ contains o. (All index arithmetic is modulo n.) Every sequence of safe vertex moves describes a homotopy between two polygons with the same number of vertices and the same winding number around the origin.

   (a) Let n be an arbitrary odd integer. Let P be a regular star polygon with n vertices spaced evenly around the unit circle, such that $wind(P, o) = \lfloor n/2 \rfloor$, as shown below. Describe how to rotate P around the origin by half a circle using $O(n)$ safe triangle moves.

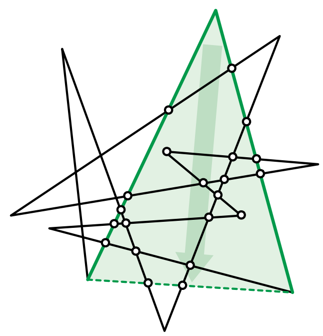


This rotation can be performed with $O(n)$ safe triangle moves.

   (b) Let P and Q be two *arbitrary* n-gons in $\mathbb{R}^2 \setminus \{o\}$, such that $wind(P, o) = wind(Q, o)$. Prove that P can be transformed into Q using $O(n)$ safe vertex moves. *[Hint: Aim for a canonical polygon with winding number* $wind(P, o)$*. The star polygon in part (a) may not be the best candidate.]*

3. Let P be an arbitrary polygon with n vertices; for simplicity, assume no three vertices in P are collinear. The *image graph* $G(P)$ is a planar straight-line graph whose nodes are the vertices of P and the intersection points of edges of P. Let N denote the number of nodes in the image graph $G(P)$. Trivially, $n \le N \le \binom{n}{2}$.

   We can easily reduce P to a triangle by repeatedly replacing two successive edges $p_{i-1}p_i$ and p_ip_{i+1} with a new edge $p_{i-1}p_{i+1}$, deleting the shared vertex p_i. Define the *cost* of deleting vertex p_i to be the number of nodes of $G(P)$ that lie in the triangle $\triangle p_{i-1}p_ip_{i+1}$, *except* for the three vertices p_{i-1}, p_i, and p_{i+1}. For example, if P is convex, every vertex deletion has cost zero.



A vertex move with cost 22.

   (a) Prove that any vertex deletion with cost k can be transformed into a sequence of $O(k)$ homotopy moves ($1 \leftrightarrow 0$, $2 \leftrightarrow 0$, and $3 \to 3$), by treating the polygon as a generic curve. *[Hint: Watch out for spurs!]* This is the motivation for my definition of cost.

   (b) Trivially, every vertex deletion has cost $O(n^2)$, and therefore any sequence of $n-3$ vertex deletions has total cost $O(n^3)$. Prove that this $O(n^3)$ bound is tight in the worst case. *[Hint: See problem 2.]*

   (c) Prove that if P is simple, there is a sequence of $n-3$ vertex deletions with total cost 0.

   (d) Prove that any polygon P can be reduced to a triangle by a sequence of $n-3$ vertex deletions with cost $O(N^2)$. *[Hint: A vertex deletion can actually increase N.]*

   ★(e) Prove or disprove: There is a sequence of $n-3$ vertex deletions with total cost $O(N^{3/2})$. Your proof of part (b) implies a worst-case $\Omega(N^{3/2})$ lower bound.