These questions are rather long, so I don’t expect everyone to submit solution to everything.

1. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be a set of generic closed curves that intersect each other only pairwise, transversely, and away from their vertices; we call any such set a \textit{(generic) family of planar curves}. For simplicity, assume that every curve $\gamma_i$ is either non-simple or intersects another curve $\gamma_j$. Then the image of $\Gamma$ is a 4-regular plane graph, which may be disconnected. Conversely, every 4-regular plane graph is the image of a family of planar curves.

A \textit{Gauss paragraph} of $\Gamma$ is a family of $k$ strings, obtained by uniquely labeling the vertices of $\Gamma$, and then listing the vertices in order along each curve $\gamma_i$ in an arbitrary direction, starting at an arbitrary basepoint, considering the curves in arbitrary order.

\begin{center}
\includegraphics[width=0.5\textwidth]{gauss_paragraph.png}
\end{center}

A connected family of planar curves with Gauss paragraph $\{acfe, jgfbadgi, iebcdhhj\}$.

Now let’s go the other direction. A Gauss paragraph is any set of non-empty strings in which any symbol appears exactly twice or not at all. Each Gauss paragraph $X$ defines a 4-regular graph $G(X)$ whose vertices are the characters in $X$ and whose edges correspond to adjacent character pairs. If $X$ is the Gauss paragraph of a family of planar curves, then $G(X)$ is the image graph of that family.

(a) Prove that if $X$ is the Gauss paragraph of a family of planar curves, then the edges of $G(X)$ can be directed so that every vertex has in-degree 2 and out-degree 2. \textit{[Hint: Consider self-intersection points of one curve and intersection points of two curves separately.]} 

(b) Describe a linear-time algorithm that either directs the edges of $G(X)$ as described in part (a) or correctly reports that no such orientation exists, given the Gauss paragraph $X$ as input.

(c) Prove that $X$ is the Gauss paragraph of a generic family of planar curves if and only if $G(X)$ has a planar embedding such that every component has a weakly simple Euler tour.

(d) Sketch a linear-time algorithm to decide whether a given string is the Gauss paragraph of a generic family of planar curves. (Just describe the necessary modifications to the algorithm for single curves.)
2. Fix an arbitrary point \( o \) in the plane, called the origin. Let \( P \) be a polygon in \( \mathbb{R}^2 \setminus o \) with vertices \( p_0, p_1, \ldots, p_{n-1} \). A vertex move on \( P \) replaces an arbitrary vertex \( p_i \) with a new point \( q_i \). This vertex move is safe if neither of the triangles \( \triangle p_i q_i p_{i-1} \) and \( \triangle p_i q_i p_{i+1} \) contains \( o \). (All index arithmetic is modulo \( n \).) Every sequence of safe vertex moves describes a homotopy between two polygons with the same number of vertices and the same winding number around the origin.

(a) Let \( n \) be an arbitrary odd integer. Let \( P \) be a regular star polygon with \( n \) vertices spaced evenly around the unit circle, such that \( \text{wind}(P, o) = \lfloor n/2 \rfloor \), as shown below. Describe how to rotate \( P \) around the origin by half a circle using \( O(n) \) safe triangle moves.

(b) Let \( P \) and \( Q \) be two arbitrary \( n \)-gons in \( \mathbb{R}^2 \setminus \{o\} \), such that \( \text{wind}(P, o) = \text{wind}(Q, o) \). Prove that \( P \) can be transformed into \( Q \) using \( O(n) \) safe vertex moves. [Hint: Aim for a canonical polygon with winding number \( \text{wind}(P, o) \). The star polygon in part (a) may not be the best candidate.]
3. Let $P$ be an arbitrary polygon with $n$ vertices; for simplicity, assume no three vertices in $P$ are collinear. The image graph $G(P)$ is a planar straight-line graph whose nodes are the vertices of $P$ and the intersection points of edges of $P$. Let $N$ denote the number of nodes in the image graph $G(P)$. Trivially, $n \leq N \leq \binom{n}{2}$.

We can easily reduce $P$ to a triangle by repeatedly replacing two successive edges $p_{i}p_{i+1}$ and $p_{i}p_{i+1}$ with a new edge $p_{i-1}p_{i+1}$, deleting the shared vertex $p_{i}$. Define the cost of deleting vertex $p_{i}$ to be the number of nodes of $G(P)$ that lie in the triangle $\triangle p_{i-1}p_{i}p_{i+1}$, except for the three vertices $p_{i-1}$, $p_{i}$, and $p_{i+1}$. For example, if $P$ is convex, every vertex deletion has cost zero.

(a) Prove that any vertex deletion with cost $k$ can be transformed into a sequence of $O(k)$ homotopy moves ($1\rightarrow 0$, $2\rightarrow 0$, and $3\rightarrow 3$), by treating the polygon as a generic curve. [Hint: Watch out for spurs!] This is the motivation for my definition of cost.

(b) Trivially, every vertex deletion has cost $O(n^2)$, and therefore any sequence of $n-3$ vertex deletions has total cost $O(n^3)$. Prove that this $O(n^3)$ bound is tight in the worst case. [Hint: See problem 2.]

(c) Prove that if $P$ is simple, there is a sequence of $n-3$ vertex deletions with total cost 0.

(d) Prove that any polygon $P$ can be reduced to a triangle by a sequence of $n-3$ vertex deletions with cost $O(N^2)$. [Hint: A vertex deletion can actually increase $N$.]

* (e) Prove or disprove: There is a sequence of $n-3$ vertex deletions with total cost $O(N^{3/2})$. Your proof of part (b) implies a worst-case $\Omega(N^{3/2})$ lower bound.