Recall that a ***polygon with holes*** is a region in the plane whose boundary consists of an outer simple polygon $P_0$ and one or more inner simple polygons $P_1, \dots, P_h$ with disjoint interiors that lie in the interior of $P_0$. The interiors of the inner polygons are called *holes*.

1. Suppose we are given a polygon with holes $\mathcal{P}$ (specified by the polygons $P_0, P_1, \dots, P_h$) and two polygons $\alpha$ and $\beta$ in $\mathcal{P}$, which may or may not be simple.

   (a) Describe an algorithm to determine whether $\alpha$ and $\beta$ are homotopic (with no fixed base point) in $\mathcal{P}$.

   (b) Describe a faster algorithm for part (a) for the special case when $\alpha$ and $\beta$ are both simple.

   (c) Describe an algorithm to compute the minimum-length polygon that is homotopic in $\mathcal{P}$ to $\alpha$.

   All three problems can be solved by modifying algorithms we've seen in class for polygonal *paths* (where any homotopy is required to keep the endpoints fixed). You don't need to describe these algorithms in complete detail; just describe the necessary modifications.

2. Let $\mathcal{P}$ be a polygon with holes, specified by the polygons $P_0, P_1, \dots, P_h$. For this problem, we define an ***arc*** in $\mathcal{P}$ to be any path in $\mathcal{P}$ whose endpoints lie on the outer polygon $P_0$. Two arcs $\alpha$ and $\beta$ are ***homotopic relative to $P_0$***, or just ***relatively homotopic***, if there is a continuous function $h\colon [0,1] \times [0,1] \to \mathcal{P}$ satisfying four conditions:

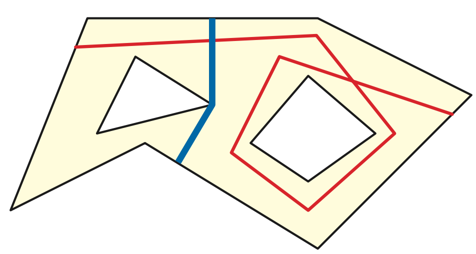
   - $h(s,0) = \alpha(s)$ and $h(s,1) = \beta(s)$ for all $s \in [0,1]$.
   - $h(0,t) \in P_0$ and $h(1,t) \in P_0$ for all $t \in [0,1]$.

   Informally, a relative homotopy deforms one arc into another through a continuous sequence of arcs.

   (a) Prove that every pair of arcs in $\mathcal{P}$ is relatively homotopic if and only if $\mathcal{P}$ has at most one hole ($h \le 1$).

   (b) Describe an algorithm to determine whether two given *polygonal* arcs $\alpha$ and $\beta$ are relatively homotopic. *[Hint: Modify the crossing-sequence algorithms for testing standard homotopy.]*

   ★(c) Describe a fast algorithm to compute the shortest arc that is relatively homotopic to a given polygonal arc.

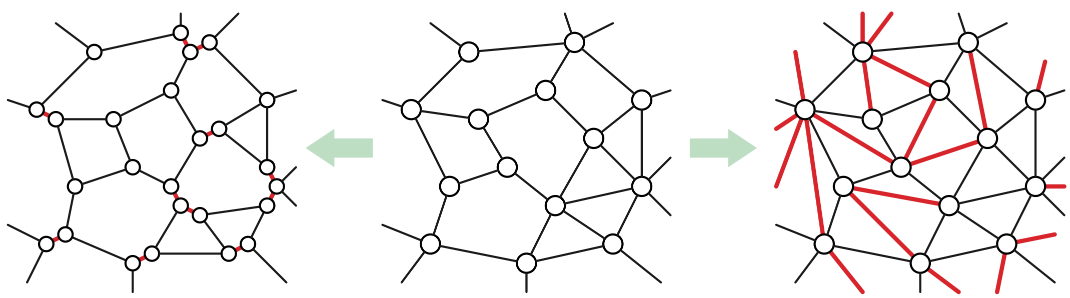


The blue (thick) arc is the shortest arc that is relatively homotopic to the red (thin) arc.

3. When we think about algorithms for planar graphs, it is sometimes useful to make assumptions about the degrees of vertices and/or the degrees of faces. Let $G$ be a simple undirected planar graph with weighted edges.

   - We can assume without loss of generality that every face of $G$ has degree 3, by inserting diagonals into any higher-degree face. Moreover, we can preserve shortest-path distances by giving these new edges infinite weight (length), and we can preserve maximum flow values by giving the new edges weight (capacity) zero.
   - We can assume without loss of generality that every vertex of $G$ has degree 3, by expanding each higher-degree vertex into a tree of degree-3 vertices. Moreover, we can preserve shortest-path distances by giving these new edges weight (length) zero, and we can preserve maximum flow values by giving the new edges infinite weight (capacity).

   Unfortunately, triangulating faces increases vertex degrees, and expanding vertices increases face degrees. What if we need both vertices and faces to have small degree?



   Prove that for some constants $\Delta$, $\Delta^*$, and $c$, any simple $n$-vertex planar graph $G$ can be modified by inserting and expanding edges into a new simple planar graph $\tilde{G}$ with $cn$ vertices, such that each vertex has degree at most $\Delta$ and each face has degree at most $\Delta^*$. More simply: Argue that without loss of generality, planar graphs have bounded vertex degrees *and* bounded face degrees. Try to keep the product $\Delta \cdot \Delta^* \cdot c$ small.