Planar Graphs

\[ G = (V, E) \]

\[ V \text{ - finite set of vertices} \]

\[ E \text{ - pairs of vertices - edges} \]

\[ \text{simple graph} \]

Graph = \((V, D, \text{rev}, \text{head})\)

\[ V \text{ - vertices} \]

\[ D \text{ - darts} \]

\[ \text{even#} \]

\[ \text{rev: } D \rightarrow D \]

\[ \text{involution: } \text{rev}(\text{rev}(d)) = d \neq \text{rev}(d) \]

\[ \text{head: } D \rightarrow V \]

Edge: \(e = \{d, \text{rev}(d)\}\)

\[ \deg(v) = \# \text{head}^{-1}(v) \]
Topological graphs

\[ V^T = \text{distinct points} \]
\[ E^T = \text{distinct real intervals} \]

\[ G = (V^T \cup E^T) / \sim \]
\[ e^T(0) \sim v^T \quad \text{if} \quad e = uv \]
\[ e^T(1) \sim u^T \]

Deletion

If \( e \) is not a bridge, \( G \setminus e \) is connected

Contraction

If \( e \) is not a loop then \( G / e \) makes sense
For all edges $e$:
- If $e$ is a loop, delete $e$.
- Else, contract $e$.

Contracted edges define a spanning tree.

Color edges arbitrarily:
- Every cycle has at least 1 red edge.
- Every cycle has at least 1 blue edge.

Blue = spanning tree.
Planar graph

$G$ is planar if there is an embedding $\phi: G \to \mathbb{R}^2$

outer face

$\text{deg}(f) = \# \text{left}_\text{right}(f)$

next: $D \to D$

encodes darts with same shore in cyclic order clockwise

$\text{succ}: D \to D$

$\text{succ} = \text{rev} \circ \text{next}$

next = $\text{rev} \circ \text{succ}$
$G$ planar map

$G^*$ dual map

$\text{deletion} = \text{contraction}^*$

$\text{cycle} = \text{cut}^*$

$\text{[whitney]}$

spanning tree $T$

acyclic

connected

$(E \setminus T)^*$

dual

connected

acyclic
**Straight-line embedding**

**Theorem**

Every simple planar graph has a straight emb.

WLOG every face is a $\triangle$ except outer simple.

\[ \text{Diagram of planar graph} \]