Homotopy

Definition: continuous deformation

A homotopy from \( \alpha: S^1 \to \mathbb{R}^2 \) to \( \beta: S^1 \to \mathbb{R}^2 \)

is a continuous function \( h: S^1 \times [0,1] \to \mathbb{R}^2 \) s.t.

\[
h(\theta, 0) = \alpha(\theta) \quad h(\theta, 1) = \beta(\theta)
\]

Intuitively, second parameter is "time"

There is a homotopy between any two planar curves:

\[
h(\theta, t) = (1-t) \alpha(\theta) + t \beta(\theta)
\]

But linear interpolation isn’t necessarily “nice”

Combinational Homotopy — Depends on curve representation

- Polygons: Triangle moves or Vertex moves

- Graph walks: Face moves and edge moves replace part of face boundary with its complement

- Generic curves: "Homotopy moves" [Steinitz, Alexandru, Briggs, Reidemeister, Tins,...]
In all models, an arbitrary homotopy between nice curves is equivalent to a finite sequence of moves.

Formally, homotopic — There is a function

\[ H : S^1 \times [0,1] \times [0,1] \rightarrow \mathbb{R}^2 \]

s.t. \( H(\cdot, \cdot, 0) = h \) and \( H(\cdot, \cdot, 1) = h' \)

Given \( \text{Finite moves} \)

Moreover, \( \| H(\theta, t, u) - h(\theta, t) \| < \varepsilon \)

for all \( \theta, t, u \)

for any desired \( \varepsilon > 0 \).

“Simplicial approximation theorem”

**Obstacles**

Suppose we want our homotopy to avoid some point in the plane. (WLOG \( c = (0,0) \).

Now not every pair of curves is homotopic

\[ h : S^1 \times [0,1] \rightarrow \mathbb{R}^2 \setminus c \]

**[Hopf 1935]**

Lemma: \( \alpha \) and \( \beta \) are homotopic curves in \( \mathbb{R}^2 \setminus c \)

if and only if \( \text{wind}(\alpha, c) = \text{wind}(\beta, c) \)

Proof:

\( \Rightarrow \) (Project \( \alpha, \beta, h \) onto \( S^1 \), argue about degree)

\( \Rightarrow \) Consider polygonal homotopy

wind \( = 1 + 1 - 1 - 1 = -3 \)

same +

same -

dec +

dec -

inc +

inc -
Exercise for polygons/generic

We say that winding # is a **homotopy invariant**

Algorithm problem: How many moves do we need?

Polygons: $O(n)$ [trivial]  
Graph walks: $O(n+k)$

Homotopy moves: $O(n^2)$ [Steinitz]

Proof: Fix your favorite curve $\gamma$, let $n=#$ vertices

If $\gamma$ is simple, nothing to do, so assume $n>0$.

Any vertex splits $\gamma$ into smaller curves

By induction, some vertex splits off a simple sub curve

If loop is empty, can be removed by $1-30$

If not empty, some strand crosses

- if strand is non-simple, recurse on simple sub curve with fewer faces

Simple strand creates a bigon

An inclusion-minimal/irreducible bigon:

All crossing strands are simple
Any pair crosses at most once
We can remove a minimal bigon with 3→3's and one 2→0

Enter ⇒ every min bigon has a triangular face on its boundary

#moves = #faces of bigon ≤ #faces of $X = n-2$ [Euler]

So we can either remove 1 vertex with 1 move
or 2 vertices with 2 moves

This is not optimal. $\Theta(n^{7/6})$ [BE17] / It is optimal in $R^2 \setminus O^1$ [BE18]

Regular homotopy / rotation #

Bradwardine (1350)
Meister (1722) - vertex moves don’t change sum of external angles except when they collapse corners

A regular polygonal homotopy avoids spurs/spikes/whiskers/0° angles

Rotation # = Ext angles is a regular homotopy invariant.

Lemma: Two polygons are reg homotopic iff same rotation #
Whitney-Graustein Theorem: Arbitrary curves are reg. homotopic \( \iff \) equal rotation.

"regular" = continuous non-zero tangents

Generic curves: Allow only \( \rightarrow 0 \) and \( 3 \rightarrow 3 \) moves.

\[ \rightarrow 0 \]
\[ \rightarrow \]

Lose one rotation

\[ \uparrow \text{ok} \]
\[ \downarrow \text{ok} \]

Theorem: Any curve can be transformed into a canonical curve with the same rotation number using only reg. homotopy moves.

\[ \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \ldots \]

Proof: By Steinitz, there is an empty loop or bigon.

Loop: slide to neighborhood of basepoint and ignore (for now)

\[ \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow 3 \rightarrow 0 \rightarrow 1 \]

Bigon: remove 2 vertices in Oh, moves

Eventually reach circle with loop

Apply "Whitney trick" to cancel opposite loops

\[ \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow 3 \rightarrow 0 \rightarrow 2 \rightarrow 0 \rightarrow \]

This is optimal. [Nowik 2014]
Curve invariants: \cite{Arnold, Arcadi, Polyak, et al.}

\[
\text{Defect} = -2 \sum_{x \neq y} \text{sgn}(x) \cdot \text{sgn}(y)
\]

\cite{Shumakovitch}

\[
\text{Strangeness} = \left( \text{wind}(1, \delta(0))^{2-1/4} \right) + \sum_{x \text{ int}} \text{sgn}(x) \cdot \text{wind}(1, x)
\]

Lemma: Defect + strangeness are independent of basept + orientation.

Lemma: Defect + strangeness change as follows:

1 \rightarrow O: Defect unchanged
2 \rightarrow O: change by -2, 0, 2
3 \rightarrow B: change by \pm 2

Lemma: Defect(0) = 0

\[
\text{Canonical curves have } S_t = 0
\]

Lemma: There are \( n \)-vertex curves with defect = \( O(n^{3/2}) \) and strangeness = \( O(n^2) \)

These are worst possible: exact constant factors.

A "grid"