Office hours: Fri 4-5 every week (this week but not next week)
No class next week — HW 1 "extension" to following week
but HW 2 coming soon!

Homotopy between curves \( \gamma, \gamma' : S^1 \to \mathbb{R}^2 \) is
\[ h : S^1 \times [0,1] \to \mathbb{R}^2 \]
\[ \text{"time"} \]
s.t. \( h(\theta,0) = \gamma(\theta) \), \( h(\theta,1) = \gamma'(\theta) \)
Equivalent: \( h : [0,1] \to (S^1 \to \mathbb{R}^2) \)
\[ h(0) = \gamma \quad h(1) = \gamma' \]

Any two curves in the plane are homotopic
\[ h(\theta,t) = (1-t) \cdot \gamma(\theta) + t \cdot \gamma'(\theta) \]

But homotopies aren't always "nice"

Polygons:
- vertex move
- triangle move

Graph walks:
- face move
- edge/spike/spur/whisker move

Generic:
\[ 1 \to 0 \quad 2 \to 1 \quad 3 \to 5 \quad \text{homotopy moves} \]

Reidemeister/Alexander–Briggs/Steinitz/Tits
Arbitrary homotopies are equivalent to finite seq. of moves. 

\[ H : S^2 \times [0,1] \times [0,1] \to \mathbb{R} \]

\[ H(0) = h_{ugly} \quad H(1) = h_{nice} \]

**Obstacles:** Suppose we want to avoid point \( o = (0,0) \)

\[ \text{Theorem [Hopf]}: \quad \text{Two curves } \alpha, \beta : S^1 \to \mathbb{R}^2 \setminus o \]

\[ \alpha \text{ and } \beta \text{ are homotopic iff } \text{wind}(\alpha, o) = \text{wind}(\beta, o) \]

**Proof sketch:**

\[ \text{wind doesn't change} \]

\[ \text{wind} = -2 \]

\[ \text{[Exercise]} \]

Winding # is \( \text{[homotopy invariant]} \)

**Algorithm ?:** How many steps?

- Polygons: \( O(n) \)
- Graph: Graph walks: \( O(n + l) \)
- Homotopy moves: \( O(n^2) \) [Steinitz 1916]
Find a simple subcurve

IF contains a smaller simple subcurve, recurse

\[ \rightarrow \text{wlog, every "strand" crossing subcurve is simple} \]

Find bigon

wlog - minimal bigon

Pseudoline arrangement

We can empty any min bigon with 3 \rightarrow 3 move per face

\[ \rightarrow \]

\( n \) moves to kill 2 vertices

so \( O(n^2) \) moves in total

\( \square \)

Not optimal: \( \Theta(n^{3/2}) \) [Cheng & E’17]

\( \mathbb{R}^2 \setminus o : \Theta(n^2) \) [Steinitz/Haas Scott] [Cheng & E’18]
Regular homotopy

Whitney

Meister [1777]

\[ R \rightarrow \mathbb{N} \rightarrow \mathbb{S} \]

M: sum of exterior angles is invariant for vertex moves iff never collapse an angle

Rotation # is a regular hom. invariant

Whitney - Graustein Theorem [Boy, 1903]

Reg hom allows

Canonical curves

Proof: Pick X

Steinitz: either empty loop \rightarrow move it to danger

minimal digon \rightarrow kill it and recurse
After $O(n^2)$ moves:

$O(n^2)$ is optimal. [Manik 2014]