Approximation via Decomposition

Max Indep Set

Parameter: branchwidth

1. MIS can be computed in \( n \cdot 2^{O(bw)} \) time
2. Any planar graph can be decomposed into subgraphs with small bw
   \[ \Rightarrow (1-\epsilon)\text{-approx MIS in } n \cdot 2^{O(bw)} \text{ time.} \]

Linear- Time Approx Scheme

Branchwidth

Branch-decomposition:
- Hierarchical clustering of edges of \( G \)
- Carving = Linear family
- Maximal family of non-crossing subsets of \( E \)

\( \uparrow \)

Can take more: \( A \cap B = A \) or \( B \) or \( \emptyset \)

For any subset \( S \subseteq E \)

\[ \text{define } \partial S = \text{vertices incident to } S \text{ and to } E \setminus S \]

\[ \text{width of branch decomp } = \max_S |\partial S| \]

branchwidth = \( \min \) width of branch decomp

\[ bw = O(\sqrt{n}) \]

Max Indep Set \((G, B)\)

Subproblem = subset \( S \subseteq B \) and ind. set \( E \in \partial S \)

\[ \# \text{subs } = (2E-1) \cdot 2^w \]

each depends on \( \leq 2^bw \) smaller subs

\[ \Rightarrow O(E \cdot 4^{bw}) = E \cdot 2^{O(bw)} \text{ time} \]

Planar: \( n \cdot 2^{O(bw)} \) time

branchwidth \((G) \leq O(\min \{ \text{diam} (G), \text{diam} (G^*) \}) \)
Radial graph $G_0$ — $V_0 = V \cup F$  
E_0 = corners

Medial graph $G^x$ — $V^x = E$  
$E^x = corners$

Every face has deg 4 “Quadrangulation”

$G_0 = (G^x) \circ$  
$G^x = (G^x)^\circ$  
$G_0 = (G^x)^*$

$\text{dist}^0(u, v) \leq 2 \text{dist}^x(u, v)$

$\text{dist}^0(F, g) \leq 2 \text{dist}^x(F, g)$

$\text{diameter}(G_0) \leq 2 \min \{ \text{diam}(G), \text{diam}(G^x) \}$

Tree-creature decomposition

$T = \text{BFS of } G_0$ with diameter $d = \text{diam}(G_0)$

$C = \text{compl. spanning tree of } G^x$

$\text{root at arbitrary leaf}$

$D = \{ \text{rooted subtrees of } C \}$

$\text{non-crossing family of subsets of } V^x = E$

For every node $e$ in $C$

$S_e = \text{subtree rooted at } e$

Fund. cycle of $T$ — length $\leq d + 1$

$S_e^f = \text{one sub-subtree}$ — length $\leq d + 2$

$S_e^{\prime\prime} = \text{two sub-subtrees}$ — length $\leq d + 3$
Decomposition

Build BFS tree rooted at $r$

$$L_i = \{ v \mid \text{dist}(r,v) = i \}$$

$$L_{jk} = \bigcup_{i \mid i \mod k \neq j} L_i$$

$$\text{diam}(G[L_{jk}]) \leq k - 1 \ast$$

$$\text{bw}(G[L_{jk}]) \leq k$$

$$\text{MIS}(G[L_{jk}]) \text{ in time } n \cdot 2^{O(k)}$$

Fix $k$. $\text{OPT} = \text{MIS}(G)$

$$\text{OPT}_j = \text{OPT} \cap L_{jk}$$

$$\max_j |\text{OPT}_j| \geq \left(1 - \frac{\epsilon}{4}\right) |\text{OPT}|$$

So... choose $k = \frac{4}{\epsilon}$

$$\text{Time} = n \cdot 2^{O(1/\epsilon)}$$

try all possible $j$

$$\text{MIS}(G[L_{jk}]) \geq \text{OPT}_j \geq (1 - \epsilon) \text{OPT}$$

Vertex Cover

Overlapping pieces

$(1 + \epsilon)$-approx

in $n \cdot 2^{O(1/\epsilon)}$ time