Reflection system $(\Phi, a,b,c)$

$\Phi = \text{flags}$  $a,b,c: \Phi \Rightarrow \Phi$ involutions

$s.t.$ $a \cdot c = c \cdot a$

Change vertex (Apex)

Change edge (Border)

Change Face (Cell)

vertices = orbits of $\langle b,c \rangle$
edges = orbits of $\langle \phi,c \rangle$
faces = orbits of $\langle a,b \rangle$

darts = orbits of $c$
corners = orbits of $b$
sides = orbits of $a$

Blades = faces of barycentric subdivision $G^+$

= vertices of band decomposition $G^\circ$

Dual reflection system: $(\overline{\phi}, e,b,c)$

$G^\circ = (G^*)^\circ$  $G^+ = (G^*)^+ = (G^\circ)^+$  $(G^\circ)^* = G^+$
Deletion: In parallel:
\[ b'(\phi) \left\{ \begin{array}{ll}
\emptyset & \text{if } b(\phi) \notin e \\
b(c(b(\phi))) & \text{if } b(c(b(\phi))) \notin e \\
b(c(b(c(b(\phi))))) & \text{otherwise}
\end{array} \right. \]
Detaching a handle by contracting a two-sided non-separating loop.

Detaching a twist by contracting a one-sided loop.

A self-dual twist.

Contracting or deleting a self-dual twist.
Classification: Every compact surface = \{ sphere with \ g \ handles \ and \ \ h \ twists \ \\
\Sigma(g, h) \}

Euler's formula: \( V - E + F = 2 - 2g - h \)

Actually: \( \Sigma(g, h) = \Sigma(g-1, h+2) \) if \( h > 0 \)

Proof (Dyck): Induction with base case \( \Sigma(1, 1) \Rightarrow \Sigma(0, 3) \)

\[ \begin{align*}
\Sigma(1, 1) & \\
\Sigma(3, 3) &
\end{align*} \]

\( \Rightarrow \) Every surface is either \( \Sigma(g, 0) \) for some \( g \geq 0 \)

or \( \Sigma(0, x) \) for some \( x > 0 \)

\( g = \text{genus} \quad 2g + h = \text{Euler genus} \)

Orientable: \( h = 0 \) \( \Leftrightarrow \) \( G^0 \) is bipartite

Discard half of the blades!

Dials of directed graphs on nonorientable surfaces don't make sense.
Surfaces with boundary
Mark some faces as deleted