Reflection system \((\Phi, a, b, c)\)

\(\Phi\) - set of blades/flags

\(a, b, c\) - involutions of \(\Phi\)

such that \(a \circ c = c \circ a\)

- vertices = orbits of \(<b, c>\)
- edges = orbits of \(<a, c>\)
- faces = orbits of \(<a, b>\)
- sides = orbits of \(a\)
- corners = orbits of \(b\)
- darts = orbits of \(c\)
Classification Theorem:

Every surface is a sphere with $g \geq 0$ handles and/or $h \geq 0$ tori attached.

Euler: $V - E + F = 2 - 2g - h$

Dyck: $\Sigma(g,h) = \Sigma(g-1,h+2)$ if $h \geq 1$
Detaching a handle by contracting a two-sided non-separating loop.

Detaching a twist by contracting a one-sided loop.

A self-dual twist.

Contracting or deleting a self-dual twist.
Surfaces with bdy

Combinatorially:
- comb. map
  - with faces marked "gone"
- disjoint!
  - $G$: holes
  - $G^*$: punctures
    - alone
    - with disk

Tree-coforest decomposition

$$(T, F, L)$$

$T =$ spanning tree of $G$

$F =$ spanning forest of $G^*$
  - with one tree per puncture

$L = E \setminus (T \cup F)$

$|L| = 2\chi_h + b - 1$

Forest-cotree decomposition

$$(\partial G, F, C, L)$$

$\partial G =$ boundary edges

$C =$ spanning tree of $G^*$
  - punctures

$F =$ spanning forest of $G$
  - with each tree containing one vertex on boundary

$L = E \setminus (\partial G \cup C \cup F)$

$|L| = 2\chi_h + b - 1$
Cut graph = subgraph of $G$ that cuts surface into a disk

Reduced = no degree-1 verts.

$\langle T, L, C \rangle$ - any tree-cotree Decomplex

$TUL = \text{cutgraph}$

Reduce by removing "hair"