Homotopy on surfaces

with boundary — easy

without boundary — harder — no fences!

Dehn (1912)

build a system of arcs

curve → crossing seq

reduce just like polygons with holes

O(gn + gl)

process

Universal cover

\[ \tilde{\gamma} \rightarrow \tilde{\gamma} \]

closed path/loop \( \gamma \) in \( \tilde{\gamma} \) is contractible

iff corresponding path \( \tilde{\gamma} \) in \( \tilde{\gamma} \) is closed

Strategy: Find a description of \( \tilde{\gamma} \)

and check if it's closed

Input: combinatorial map \( \Sigma = (V, E, F) \) — complexity \( n \)

closed walk \( \gamma \) in \( (V, E) \) — length \( l \)

Output: Is \( \gamma \) contractible in \( \Sigma \)?

Reduce \( \Sigma \) to a system of loops

\( (T, L, C) \) — tree-cotree decomp.

• contract every edge in \( T \) — also contract in \( \gamma \)

• delete every edge in \( C \) — re-route \( \gamma \) around face

• left with 2g edges \( L \) — loops + isthmuses
Result:

- System of loops - complexity $O(g)$
- Closed walk with length $O(g^2)$

Encoding: $abcddabbbc...$

Two equivalences:
- $spurs: \overline{ab} = \overline{ba} = \varepsilon$
- $face: abcd\overline{abc} = \varepsilon$
  \[ cd\overline{ab}c = \overline{b}a \overline{d} \]

Dehn’s lemma — some equivalence shortens the string (if it's contractible)

$g = 1$

\[
\begin{align*}
\text{bba} & \text{baba} \overline{b} \overline{b} \text{baba} \overline{b} \overline{b} \text{baba} \\
\text{abab} & = \varepsilon \\
\text{ab} & = \text{ba} \\
\text{ab} & = \overline{ba} \\
\text{contractible} \iff \#a = 0 \quad \#b = 0
\end{align*}
\]

$g > 1$

Universal cover

Regular tiling of the hyperbolic plane
Isoperimetric inequality

\[
\begin{align*}
\text{Euclidean:} & \quad \text{length} l \Rightarrow \text{Area} \leq O(l^2) \\
\text{Hyperbolic:} & \quad \text{Area} \leq O(l) \\
\text{Circle of radius } r \text{ Euclidean: perimeter } & \text{area } \frac{\pi r^2}{\infty} \\
\text{Hyperbolic: perimeter } & \text{area } \Theta(c^r) \\
\text{Nonempty} & \text{Dehn's lemma: Any closed walk in hyperbolic tiling contains a spur or strict majority of a tile boundary} \\
\text{Proof: WLOG assume walk is simple} & \\
\end{align*}
\]

Consider a disk in tiling D

Assign an exterior angle \( \angle_c \) to each corner of D

Define curvature of f

\[
\kappa(f) = 1 - \frac{\sum \angle_c}{c \cdot \angle f}
\]

Define curvature of \( u \)

\[
\kappa(u) = 1 - \frac{1}{2} \text{deg}(u) + \frac{\sum \angle_c}{c \cdot \text{dev}}
\]

Discrete Gauss-Bonnet

\[
\sum \kappa(f) + \sum \kappa(u) = \chi = 1
\]

F = 2 \cdot \angle c, \quad V - E + \angle 4c
Set \( c = \frac{1}{4} \) at every corner.

\[
\kappa(F) = 1 - 4g(\frac{1}{4}) = 1 - g < 0
\]

\[
\kappa(v) = \begin{cases} 
\leq 0 & \text{if int interior} \\
\frac{3}{4} & \text{if int boundary, one int corner} \\
\leq 0 & \text{other boundary}
\end{cases}
\]

\[
\Rightarrow F(1-g) + \frac{V_+}{4} \geq 1
\]

\[V_+ \geq 4(g-1) F + 1\]

Some face has \( \geq 4g - 3 \) convex vertices must be consecutive

\[\Rightarrow 4g - 2 \text{ consecutive boundary edges}\]

Algorithm: Scan thru edge seq, reduce when possible

Naïvely: check \( \Theta(g) \) strings of length \( g^{\frac{1}{2}} \) at every char.

\[\Rightarrow O(g^3) \text{ time}\]

DFA - \( O(g^2) \) prep,

\[O(1) \text{ time per char.} \Rightarrow O(gl) \text{ time}\]
\( O(n + d) \) time \([\text{Rivat-Lazars 13} / \text{Ewittley 14}]\)

System of loop \( \rightarrow \) face is too big

Instead reduce to radial graph of sys. of loops

**system of quads**

- **Bracket**

  - subpath with turn sequence
    
    \[ 1 \ 2^* \ 1 \]
    
    \[ -1 \ 0 \ 2^* \ -1 \]

  
\[ 1 \ 2 \ 2 \ 2 \ 2 \]

\[ = 12^6 1 \]

\( \text{(1) Every closed walk in quad tiling contains a spur or bracket} \)

\( \text{(2) Run-length encoding} \rightarrow \text{spend } O(1) \text{ amortized time per edge of } G \)