Homology with real coefficients:

Cycle space $\mathbb{R}^{E-V+1}$ — elements are circulations

$\phi : D \rightarrow \mathbb{R} \Rightarrow \exists \phi : V \rightarrow \mathbb{R}$ defined as

$s.t. \phi(u) = -\phi(u \rightarrow v)$

$\exists \phi(u) = \sum_{v \rightarrow u} \phi(u \rightarrow v)$

$\phi$ is a circulation iff $\exists \phi \equiv 0$

Basis: Fundamental cycles w.r.t. spanning tree

Cut space $\mathbb{R}^{F-1}$ — elements are fractional cuts

Basis: Fundamental cuts w.r.t. spanning tree

$Z$-chain/Alexander rings/face potentials $\alpha : F \rightarrow \mathbb{R}$

boundary $\partial \alpha : D \rightarrow \mathbb{R}$ defined as

$\alpha(d) = \alpha(right(d)) - \alpha(left(d))$

$\phi$ is a boundary circulation $\Leftrightarrow \exists \alpha$ s.t. $\exists \phi \equiv \alpha$

Boundary space $\mathbb{R}^{F-1}$

Two circulations are homologous if $\exists \phi = \exists \phi'$

$\Leftrightarrow \exists (\phi - \phi') \equiv 0$

$\Leftrightarrow \phi - \phi'$ is a boundary

Homology space/group $H_4(E, \mathbb{R}) \cong \mathbb{R}^{(E-V+1)-(F-1)} = \mathbb{R} \mathbb{Z}$
Basis: \( \Gamma = \{ \text{cycle}(T,e) \mid e \in E \} \) for any tree-cotree \((T,L,C)\).

Every circulation is homologous with \( \sum_{e \in \Gamma} \phi(e) \cdot [e] \) for some \( \phi(e) \in \mathbb{R}^L \).

Dual basis: \( \Delta = \{ \text{cocycle}(C,e) \mid e \in L \} \) directed arbitrarily.

\( [u-v]_i = \begin{cases} 1 & \text{if } u-v \in C_i \in \Delta_i \\ 0 & \text{if } u-v \notin C_i \end{cases} \)

\[ \sum_{e \in \Gamma} [e] = [\Delta] \text{ homology class of } \Delta \]

\[ [\phi] = \sum_{e} \phi(e) \cdot [e] \text{ homology class of } \phi \]

- \( \phi(w) = -\phi(xw(a)) \) and \( [d] = -[\text{rev}(d)] \)
- So extra factor of \( \mathbb{Z} \) whatever

\( \phi \) is bdry iff \( [\phi] = 0 \)

\( \phi \) and \( \phi' \) homologous iff \( [\phi] = [\phi'] \)

Recall from planar flows:

Given a (not necessarily feasible) flow \( F: E \to \mathbb{R} \), find a feasible flow with the same homology class.

Residual capacity \( C_f(u-v) = C(u-v) - F(u-v) \)

Now find a feasible boundary circulation in the dual residual graph \( C_f \).

Claim: