Finding Trivial Closed Walks in Directed Surface Graphs

- Shortest homotopic paths well-defined unless
  - contractible
  - anti-homotopic neg cycle pair

**Given** dir. graph \( G = (V,E) \)
embedded on some orientable surface \( \text{genus } g = O(n) \)

Is any closed walk in \( G \) contractible?
null-homologous?
ccw \( O(n) \)
bcw \( O(gn) \)

"cycle" \( \Rightarrow \) NP-hard [Cabello]

No contractible loops \( \circ \)
anti-homotopic edges \( \circ \)

**CCW in \( O(n) \) time**

- Remove any cocycles from \( G \)
- For every face \( F \)
  - if \( F \) is a disk and \( F \) is coherent
  - return True
- return False

\( f_1 \uparrow f_2 \uparrow f_3 \uparrow \cdots \uparrow f_k \uparrow f_1 \)
\( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow v_1 \)
Lemma: If $G$ has a ccw, then $G$ has a weakly simple ccw arbitrarily close to a simple cycle.

Proof: Let $W$ be any ccw.

Case 1: $W$ is simple, done.

[Hass-Scott]: There must be a monogon or a bigon.

Case 2: Follow a dual walk $FFFF...$ inside $A$.

Case 3: Get stuck at face $F$ with $cw$ boundary.

Proof: $W$ has connected interior $A$.

Lemma: Weakly simple ccw uses each edge at most once.

Proof: [Epstein 66] $w$ is boundary of a disk.

Label faces of $G$ in or out.

$\Rightarrow$ length $\leq \Theta(n)$.

Lemma: If $G$ has a weakly simple ccw, then $G$ has a simple and no cocycles.

Proof: Follow a dual walk $FFFF...$ inside $A$.

Get stuck at face $F$ with $cw$ boundary.

If $F$ is a disk, done.

Otherwise $F \subset A \Rightarrow F$ is a disk with holes.

$W = \text{bdry of any hole}$.

$W$ is contractible, encloses fewer faces than $W$. 


Lemma: No bvw uses any edge in any cocycle

\[ \alpha: F \to \mathbb{N} \] "Z-chain"

\[ \delta \alpha: E \to \mathbb{Z}, \quad \delta \alpha(e) = \alpha(\text{right}(e)) - \alpha(\text{left}(e)) \]

**Want:**
- Non-negative
- Positive edges is connected

Boundary closed walk = Euler tour of pos. cycle circulation with connected support

**WLOG**
1. No cocycles - \( G^* \) is a deg
2. \( G \) is strongly connected

\( G_0 \) = any strong comp of \( G \)
\( H_0 = G_0 \setminus \text{cocycles} \)

\( G_1 = \text{any strong comp of } H_0 \)
\( H_1 = G_1 \setminus \text{cocycles} \)

\[ \vdots \]

until \( G_i = G_{i+1} \)

\[ \Rightarrow O(n^2) \text{ time} \]

1. If \( G_i \) has more than one vertex then \( G_i \) has bvw
2. Converges after \( \leq 2g \) iterations