10 Tree-Cotree Decompositions

10.1 Important graph definitions (yawn)

We need to establish definitions for a few important structures in graphs. Most of these are likely already familiar; I recommend using this list as later reference rather than reading it as text.

walk: A sequence \( \langle s, d_1, d_2, \ldots, d_k, t \rangle \) where \( s \) and \( t \) are vertices and each \( d_i \) is a dart, such that \( \text{tail}(d_1) = s \) and \( \text{head}(d_1) = t \) and \( \text{head}(d_i) = \text{tail}(d_{i+1}) \) for each index \( i \)

length of a walk: The number of darts in the walk. The walk \( \langle s, d_1, \ldots, d_k, t \rangle \) has length \( k \).

trivial walk: A walk \( \langle s, s \rangle \) with length 0.

closed walk: A walk \( \langle s, d_2, \ldots, d_k, t \rangle \) such that \( s = t \).

open walk: A walk that is either trivial or not closed

walk from \( s \) to \( t \): A walk with specified initial vertex \( s \) and final vertex \( t \)

\( s \) can reach \( t \): There is a walk from \( s \) to \( t \). This is an equivalence relation.

component: An equivalence class for “can reach”

connected graph: A graph with exactly one component

simple walk: A walk \( \langle s, d_1, \ldots, d_k, t \rangle \) such that each vertex is the head of at most one dart \( d_i \).

path: A simple open walk, or the subgraph induced by a simple open walk

even subgraph: A subgraph in which every vertex has even degree.

cycle: A simple non-trivial closed walk, or the subgraph induced by such a walk. A minimal non-empty even subgraph.

loop: A cycle with length 1. An edge whose endpoints coincide.

cut: A partition of the vertices \( V \) into two subsets \( S \) and \( V \setminus S \)

boundary of a cut: All edges with one endpoint in \( S \) and the other in \( V \setminus S \), for some cut \((S, V \setminus S)\)

bond: A minimal nonempty edge cut

bridge: An edge cut containing a single edge

acyclic graph: A graph containing no cycles

10.2 Deletion and Contraction

Let \( G \) be an arbitrary (not necessarily planar) abstract graph with \( n \) vertices and \( m \) edges. Deleting an edge \( e \) from \( G \) yields a smaller graph \( G \setminus e \) with \( n \) vertices and \( m-1 \) edges. We also write \( G \setminus v \) to denote the graph obtained from \( G \) by deleting a vertex \( v \) and all its incident edges. Deleting a bridge disconnects the graph.

If \( e \) is not a loop, then contracting \( e \) merges the endpoints of \( e \) into a single vertex and destroys the edge, yielding a smaller graph \( G/e \) with \( n-1 \) vertices and \( m-1 \) edges. Contracting a loop is simply forbidden by definition. Contracting a loop is not (yet) defined. If \( G \) contains edges parallel to \( e \), those edges survive in \( G/e \) as loops.

More generally, deleting any subset of edges \( E' \subseteq E \) that does not contain a bond yields a connected proper subgraph \( G \setminus E' \). More generally, contracting any subset of edges \( E' \subseteq E \) that does not contain a cycle yields a proper minor \( G/E' \).

A subgraph of a graph \( G \) is another graph obtained from \( G \) by deleting edges and vertices; a proper subgraph of \( G \) is any subgraph other than \( G \) itself. (We often equate subgraphs of \( G \) with subsets of the edges of \( G \).) A minor of \( G \) is any graph obtained from a subgraph of \( G \) by contracting edges; a proper minor of \( G \) is any minor other than \( G \) itself.
10.3 Spanning trees

A spanning tree of a graph $G$ is a connected, acyclic subgraph of $G$ (more succinctly, a subtree of $G$) that includes every vertex of $G$. We leave the following lemma as an exercise for the reader.

**Lemma:** Let $G$ be a connected graph, and let $e$ be an edge of $G$.

- If $e$ is a loop, then every spanning tree of $G$ excludes $e$.
- If $e$ is not a loop, then for any spanning tree $T$ of $G/e$, the subgraph $T \cup e$ is a spanning tree of $G$.
- If $e$ is a bridge, then every spanning tree of $G$ includes $e$.
- If $e$ is not a bridge, then every spanning tree of $G \setminus e$ is also a spanning tree of $G$.

This lemma immediately suggests the following general strategy to compute a spanning tree of any connected graph: For each edge $e$, either contract $e$ or delete $e$. Loops must be deleted and bridges must be contracted; otherwise, the decision to contract or delete is arbitrary. The previous lemma implies by induction that the set of contracted edges is a spanning tree of $G$, regardless of the order that edges are visited, or which non-loop non-bridge edges are deleted or contracted.

In practice, algorithms that compute spanning trees do not actually contract or delete edges; rather, they simply label the edges as belonging to the spanning tree or not. In this context, the previous lemma can be rewritten as follows:

**Spanning Tree Lemma:** Let $G$ be a connected graph.

- Each spanning tree of $G$ excludes at least one edge from each cycle in $G$.
- For every edge $e$ of every cycle of $G$, there is a spanning tree of $G$ that excludes $e$. 


• Every spanning tree of $G$ includes at least one edge from each bond in $G$.

• For every edge $e$ of every bond of $G$, there is a spanning tree of $G$ that includes $e$.

Corollary: If the edges of a connected graph $G$ are arbitrarily colored red or blue, so that each cycle in $G$ has at least one red edge and each bond in $G$ has at least one blue edge, then the subgraph of blue edges is a spanning tree of $G$.

Given a connected graph with $n$ vertices and $m$ edges, we can compute a spanning tree in $O(n+m)$ time using depth-first search or breadth-first search. Both of these algorithms can be seen as variants of the red-blue coloring algorithm, where the order in which edges are colored is determined on the fly as they explore the graph.

10.4 Deletion and Contraction in Planar Maps

Contraction and deletion play complementary roles in planar maps. For example, contracting any (non-loop) edge identifies its two endpoints; deleting any (non-bridge) edge merges its two shores. This resemblance is not merely incidental; in fact, contraction and deletion are dual operations. Contracting an edge in any map $\Sigma$ is equivalent to deleting the corresponding edge in $\Sigma^*$ and vice versa.

Hopefully this duality is intuitively clear, but we can make it formally trivial by describing how deletion and contraction are implemented in planar maps. Let $\text{succ}$ denote the rotation system of a planar map $\Sigma$, and let $\text{succ}^* = \text{rev} \circ \text{succ}$ denote its dual rotation system, which is also the rotation system of the dual map $\Sigma^*$. Fix an arbitrary edge $e$ of $\Sigma$.

**Deletion:** Suppose $e$ is not a bridge. Then $\Sigma \setminus e$ is a planar map that contains every dart in $\Sigma$ except $e^+$ and $e^-$. Let $\text{succ} \setminus e$ and $\text{succ}^* \setminus e$ denote the induced primal and dual rotation systems of $\Sigma \setminus e$. Then for any dart $d$ in $\Sigma \setminus e$, we have

$$(\text{succ} \setminus e)(d) = \begin{cases} 
\text{succ}(\text{succ}(\text{succ}(d))) & \text{if succ}(d) \in e \text{ and } \text{succ}(\text{succ}(d)) \in e, \\
\text{succ}(\text{succ}(d)) & \text{if succ}(d) \in e, \\
\text{succ}(d) & \text{otherwise}.
\end{cases}$$

The first case occurs when $e$ is an empty loop based at the head of $d$. See the figure below. In other words, to find the successor of $d$ in $\Sigma \setminus e$, we repeatedly follow successor pointers until we reach a dart that is not in the deleted edge $e$.

It follows that the dual rotation system changes as follows:

$$(\text{succ}^* \setminus e)(d) = \begin{cases} 
\text{succ}^*(\text{succ}(\text{succ}(d))) & \text{if succ}(d) \in e \text{ and } \text{succ}^*(\text{succ}(d)) \in e, \\
\text{succ}^*(\text{succ}(d)) & \text{if succ}(d) \in e, \\
\text{succ}^*(d) & \text{otherwise}.
\end{cases}$$

**Contraction:** Suppose $e$ is not a loop. Then $\Sigma/e$ is a planar map that contains every dart in $\Sigma$ except $e^+$ and $e^-$. Let $\text{succ}/e$ and $\text{succ}^*/e$ respectively denote the induced primal and dual rotation systems of $\Sigma/e$. Then for any dart $d$ of $\Sigma/e$, we have

$$(\text{succ}^*/e)(d) = \begin{cases} 
\text{succ}^*(\text{succ}^*(\text{succ}^*(d))) & \text{if succ}^*(d) \in e \text{ and } \text{succ}^*(\text{succ}^*(d)) \in e, \\
\text{succ}^*(\text{succ}^*(d)) & \text{if succ}^*(d) \in e, \\
\text{succ}^*(d) & \text{otherwise}.
\end{cases}$$
The first case occurs when one endpoint of $e$ has degree 1 and the head of $d$ is the other endpoint of $e$. See the figure below. In other words, to find the dual successor of $d$ in $\Sigma/e$, we chase dual successor pointers until we reach a dart that is not in the contracted edge $e$.

It follows that the primal rotation system changes as follows:

$$\text{(succ/e)(d)} = \begin{cases} \text{succ(succ}^*(\text{succ}^*(d))) & \text{if succ(d) \in e and succ}^*(\text{succ}(d)) \in e, \\ \text{succ(succ}^*(d)) & \text{if succ(d) \in e,} \\ \text{succ(d)} & \text{otherwise.} \end{cases}$$

Both of these formulas are trivially correct when we either delete or contract the only edge in a one-edge map, because the resulting trivial map has no darts. Assuming standard data structures, any edge can be contracted or deleted in $O(1)$ time.

The following lemma is now purely mechanical.

**Lemma (contraction ⇔ deletion):** Fix a planar map $\Sigma$, and let $e$ be any edge in $\Sigma$.

(a) If $e$ is not a loop, then $e^*$ is not a bridge and $(\Sigma/e)^* = \Sigma^* \setminus e^*$.

(b) If $e$ is not a bridge, then $e^*$ is not a loop and $(\Sigma \setminus e)^* = \Sigma^*/e^*$.

If we delete a bridge using the formulas above, the components of $G \setminus e$ become embedded independently, each on its own plane/sphere; instead of merging two faces into one, the deletion breaks one face (on either side of the deleted edge) into two. Symmetrically, if we contract a loop using the formula above, instead of merging two vertices into one, we split the single endpoint of the loop into two, splitting the graph into two independent subgraphs, one “inside” the loop and the other “outside”.

**10.5 Tree-Cotree Decompositions**

**Lemma (even subgraph ⇔ edge cut):** Fix a planar map $\Sigma$. A subset $H$ of the edges of $\Sigma$ is an even subgraph if and only if the corresponding subset $H^*$ of edges in $\Sigma^*$ is an edge cut.

**Proof:** Let $H$ be an even subgraph of $\Sigma$. Let $C_1, C_2, \ldots, C_k$ be edge-disjoint cycles in $\Sigma$ whose union is $H$. Color each vertex of $\Sigma^*$ black if it lies in the interior of an odd number of cycles $C_i$, and white otherwise. Then $H$ is the subgraph of edges with one white shore and
one black shore. It follows that $H^*$ is the subgraph of dual edges with one endpoint of each color; in other words, $H^*$ is an edge cut in $\Sigma^*$.

On the other hand, let $H^*$ be an edge cut in $\Sigma^*$. Then it is possible to color the vertices of $\Sigma^*$ black and white, so that $H^*$ is the subset of edges with one white endpoint and one black endpoint. The primal subgraph $H$ contains precisely the edges of $\Sigma$ with one white shore and one black shore. Every vertex of $\Sigma$ is incident to an even number of such edges. We conclude that $H$ is an even subgraph of $\Sigma$.

**Corollary (cycle ⇔ bond):** A subgraph $H$ of a planar map $\Sigma$ is a cycle if and only if the corresponding subgraph $H^*$ of $\Sigma^*$ is a bond.

**Proof:** A cycle is a minimal non-empty even subgraph; a bond is a minimal non-empty edge cut.

Equivalently, a cycle is a minimal subset of edges that cannot all be contracted, and a bond is a minimal subset of edges that cannot all be deleted.

**Corollary (spanning tree ⇔ spanning cotree):** Fix a planar map $\Sigma = (V, E, F)$, and let $T \sqcup C$ be a partition of $E$. Then $T$ defines a spanning tree of $\Sigma$ if and only if $C^* \subset E^*$ defines a spanning tree of $\Sigma^*$.

**Proof:** Let $T$ be an arbitrary spanning tree of $G$, and let $C^* = E^* \setminus T^*$ be the complementary dual subgraph of $\Sigma^*$. The Spanning Tree Lemma implies that every cycle of $\Sigma$ excludes at least one edge in $T$, and every bond of $\Sigma$ contains at least one edge in $T$. Cycle-bond duality implies that every bond of $\Sigma^*$ contains at least one edge in $C^*$, and every cycle of $\Sigma^*$ excludes at least one edge in $C^*$. We conclude that $C^*$ is a connected acyclic spanning subgraph of $\Sigma^*$, or in other words, a spanning tree of $\Sigma^*$.

![Figure 5: A tree-cotree decomposition of a planar map and its dual.](image)

The partition $T \sqcup C$ of edges of a planar map into primal and dual spanning trees is called a *tree-cotree decomposition*. Notice that either the primal spanning tree $T$ or the dual spanning tree $C^*$ can be chosen arbitrarily.

The duality between cycles and bonds was first proved by Hassler Whitney. Whitney also proved the following converse result. An *algebraic dual* of an abstract graph $G$ is another abstract graph $G^*$ with the same set of edges, such that a subset of edges defines a cycle in $G$ if and only if the same subset defines a bond in $G^*$.

**Theorem (Whitney (1932)):** A connected abstract graph is planar if and only if it has an algebraic dual.
10.6 Euler’s Formula

Arguably the earliest fundamental result in combinatorial topology is a simple formula first published by Leonhard Euler, but described in full generality over a century earlier by René Descartes, and described for the special case of Platonic solids by Francesco Maurolico a century before Descartes. I’ll provide two short proofs here, one directly inductive, the other relying on tree-cotree decompositions.

Euler’s Formula for Planar Maps. For any connected planar map with $n$ vertices, $m$ edges, and $f$ faces, we have $n - m + f = 2$.

Proof (by induction): Fix an arbitrary planar map $\Sigma$ with $n$ vertices, $m$ edges, and $f$ faces. If $\Sigma$ has no edges, it has one vertex and one face. Otherwise, let $e$ be any edge of $\Sigma$; there are two overlapping cases to consider.

- If $e$ is not a bridge, then deleting $e$ yields a planar map $\Sigma \setminus e$ with $n - 1$ vertices, $m - 1$ edges, and $f - 1$ faces. The induction hypothesis implies that $n - (m - 1) + (f - 1) = 2$.
- If $e$ is not a loop, then contracting $e$ yields a planar map $\Sigma / e$ with $n - 1$ vertices, $m - 1$ edges, and $f$ faces. The induction hypothesis implies that $(n - 1) - (m - 1) + f = 2$.

In all cases, we conclude that $n - m + f = 2$.

Proof (von Staudt 1847): Fix an arbitrary planar map $\Sigma$ with $n$ vertices, $m$ edges, and $f$ faces. Let $T$ be an arbitrary spanning tree of $\Sigma$. Because $T$ has $n$ vertices, it also has $n - 1$ edges. The complementary dual subgraph $C^* = (E \setminus T)^*$ is a spanning tree of $\Sigma^*$. Because $C^*$ has $f$ vertices, it also has $f - 1$ edges. Every edge in $\Sigma$ is either an edge of $T$ or the dual of an edge in $C^*$, but not both. We conclude that $m = (n - 1) + (f - 1)$.

There are many many other proofs of Euler’s formula. David Eppstein has a web page describing twenty of them, but even David’s list is incomplete. For example, we can leverage our earlier proof of Euler’s formula for planar curves, after establishing a few additional definitions.

A corner of a planar map $\Sigma$ is an incidence between a face and a vertex. (Formally, a corner is just a nickname for a dart; for each dart $d$, the corresponding corner is the incidence between the vertex $\text{head}(d)$ and the face $\text{left}(d)$, or equivalently, between the vertex $\text{tail}(\text{succ}^*(d))$ and the face right($\text{succ}(d)$).)

The medial map $\Sigma^\times$ of a planar map $\Sigma$ is another planar map whose vertices correspond to edges of $\Sigma$, whose edges correspond to corners of $\Sigma$, and whose faces correspond to vertices and faces of $\Sigma$. Specifically, two vertices are connected by an edge in $\Sigma^\times$ if the corresponding edges in $\Sigma$ are adjacent in cyclic order around any vertex (or equivalently, around any face). This definition excludes the trivial map $\Sigma$ with one vertex, no edges, and one face; the most sensible definition of $\Sigma^\times$ in that case is a simple closed curve (with two faces). Every map $\Sigma$ and its dual $\Sigma^*$ share the same medial map $\Sigma^\times$.

Every medial map $\Sigma^\times$ is either a simple cycle or 4-regular, and therefore is the image map of a connected planar multicurve. (Steinitz used medial maps (“$\Theta$-Prozeß”) to reduce his eponymous theorem about graphs of convex polyhedra to an argument about curves.)

Proof (via medial homotopy): Fix an arbitrary planar map $\Sigma$ with $n$ vertices, $m$ edges, and $f$ faces. The medial map $\Sigma^\times$ is the image map of a connected planar multicurve with $2m$
vertices and \( n + f \) faces. We already proved by induction\(^1\) that every connected planar multicurve with \( N \) vertices has exactly \( N + 2 \) faces. We conclude that \( n + f = 2m + 2 \).

(Contraction and deletion in \( \Sigma \) is equivalent to two different smoothing operations on the medial map \( \Sigma^* \). I suppose it might have been easier to prove the theorem for curves using smoothing instead of homotopy moves, but then we'd just have our first inductive proof for planar maps again!)

Later we will see yet another proof of Euler's formula (not on David’s list) based on Schnyder woods.

### 10.7 The Combinatorial Gauss-Bonnet Theorem

I'll close this lecture by proving a powerful reformulation of Euler's formula.

Suppose we assign a value \( \angle c \) to each corner \( c \) of a planar map \( \Sigma \), called the **exterior angle** at \( c \). Intuitively, you should think of \( \angle c \) as the signed angle between the tangent vectors to two darts \( d \) and \( \text{succ}^i(d) \) at their common endpoint \( \text{head}(d) \), but in fact \( \angle c \) can be any real (or complex!) number. As usual, we measure angles in units of circles (or “turns”), as God intended.

We can then define the **combinatorial curvature** of a face \( f \) or a vertex \( v \), with respect to this angle assignment, as follows:

\[
\kappa(f) := 1 - \sum_{c \in f} \angle c \\
\kappa(v) := 1 - \frac{1}{2} \deg(v) + \sum_{c \in v} \angle c
\]

Or more formally, equating corners with darts:

\[
\kappa(f) := 1 - \sum_{d: \text{left}(d)=f} \angle d \\
\kappa(v) := 1 - \sum_{d: \text{head}(d)=v} \left( \frac{1}{2} - \angle d \right)
\]

For example, suppose every edge of \( \Sigma \) is a line segment, and we actually measure corner angles geometrically. Then every vertex has curvature 0 (because its interior corner angles sum to one circle) and every bounded face of \( \Sigma \) has curvature 0 (because its total turning angle is 1).

\(^1\)Well, okay, we only proved this formula for curves, but extending our inductive proof to multicurves requires us to consider only one additional case. Suppose some \( 2 \to 0 \) move disconnects a multicurve \( \Gamma \) into two smaller connected multicurves \( \Gamma' \) and \( \Gamma'' \). The original map \( \Gamma \) has \( n^i + n^o + 2 \) vertices and \( f^i + f'^i \) faces (including the deleted bigon and the common outer face), and the induction hypothesis implies that \( f^i = n^i + 2 \) and \( f'^i = n^o + 2 \).
However, the outer face is oriented clockwise instead of counterclockwise, so its total turning angle is $-1$, and thus its curvature is 2. That 2 is actually the same as the 2 in Euler’s formula.

Alternatively, suppose $\Sigma$ is actually embedded on the unit sphere, every edge is an arc of a great circle, and angles are again measured geometrically (between tangent vectors). Then every vertex of $\Sigma$ has curvature zero, because interior angles sum to one circle, and a bit of spherical trigonometry implies that every face of $\Sigma$ has curvature equal to its area divided by $2\pi$. Because the unit sphere has surface area $4\pi$, the sum of all the face curvatures is 2. That 2 is actually the same as the 2 in Euler’s formula! (In fact, this is how Lagrange actually proved Euler’s formula for the first time.)

**The Combinatorial Gauss-Bonnet Theorem:** For any planar map $\Sigma = (V, E, F)$ and for any assignment of angles to the corners of $\Sigma$, we have $\sum_{v \in V} \kappa(v) + \sum_{f \in F} \kappa(f) = 2$.

**Proof:** We immediately have $\sum_{f} \kappa(f) = |F| - \sum_{c} \angle c$ and $\sum_{v} \kappa(f) = |V| - |E| + \sum_{c} \angle c$, which implies that $\sum_{v} \kappa(v) + \sum_{f} \kappa(f) = |V| - |E| + |F| = 2$ by Euler’s formula.

As a final geometric example, suppose $\Sigma$ is actually the complex of vertices, edges, and faces of a three-dimensional convex polyhedron (which is homeomorphic to a sphere), and again, angles are measured geometrically. Each face of $\Sigma$ is a convex planar polygon, and therefore has curvature zero. The interior angles at each vertex of $\Sigma$ sum to less than a full circle, so every vertex has positive curvature. The Combinatorial Gauss-Bonnet Theorem implies that the sum of the vertex curvatures is exactly 2. In other words, the sum of the angle defects at the vertices is two full circles, or eight right angles. Descartes described this angle defect theorem, and derived Euler’s formula (for convex polyhedra) from it, in his unpublished note *Progymnasmata de solidorum elementis* [*Exercises in the Elements of Solids*], which he most likely wrote around 1630.

### 10.8 Aptly Named

- Outerplanar graphs/maps
- Easy consequences of Euler’s formula:
  - Every simple planar graph has a vertex of degree at most 5.
  - Every planar triangulation has $3n - 6$ edges and $2n - 4$ faces.
  - Every simple planar graph has at most $3n - 6$ edges.
  - Every simple planar bipartite graph has at most $2n - 4$ edges.
  - Every planar map has either a vertex or a face of degree at most 3.
  - $K_{3,3}$ and $K_5$ are not planar.
  - There are only five Platonic solids.
  - Every loop-free planar graph is 6-colorable.

---

2It is a matter of surprisingly intense scholarly dispute whether Descartes actually stated Euler's formula, and therefore deserves to share credit with Euler, or only came close, and therefore does not. The result that Descartes actually proves is "Ponam semper pro numero angulorum solidorum $a$ & pro numero facirum $\varphi$ . . .. Numerus verorum angulorum planorum est $2\varphi - 2a - 4$. [I always write $a$ for the number of solid angles and $\varphi$ for the number of faces... The total number of plane angles is $2\varphi - 2a - 4$.]" In modern terminology, "solid angles" are vertices and "plane angles" are corners (or darts). Elsewhere in *Progymnasmata*, Descartes observed that the number of plane angles is exactly twice the number of vertices, and he used the numbers of vertices, edges, and faces of the Platonic and several Archimedean solids to derive formulas for corresponding figurate numbers. Descartes did not express his formula using the syntax $V - E + F = 2$, but in my opinion, this is entirely a matter of notational emphasis, not content or understanding. Had Descartes actually published his *Progymnasmata*, I believe even Euler (who exhibited surprise that the formula was not already known) would have called it "Descartes' formula".
– Every planar graph has an independent set of size \( \Omega(n) \) in which every vertex has degree \( O(1) \).

• Minimum spanning trees:
  – Tarjan’s red-blue meta-algorithm
  – Borůvka’s algorithm
  – Mareš’s algorithm, Matsui’s algorithm
  – minimum spanning tree ⇔ maximum spanning cotree

• Equivalence of tree-cotree decompositions and tree-onion figures

• Random (rooted) planar maps via random tree-onion figures