One-Dimensional Computational Topology Exercises

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About These Exercises

These are meant entirely for your own practice and understanding; they will not be graded. I strongly encouraged everyone to discuss these problems (and any related questions that you come up with) on the course discussion forum.

Open problems are labeled (!!). Problems that I don’t know how to solve, but that I’m not confident are open, are labeled (?). Labeled problems may or may not be difficult or interesting. Some exercises stray outside recent class material, either into the future or outside the class entirely; feel free to ask for clarification on Campuswire.

Polygons

1. Variations on polygon triangulation

In class we saw a proof, originally due to Dehn and Lennes, that the interior of any simple polygon in the plane can be triangulated by adding interior diagonals. The proof relies on the Jordan curve theorem. In this problem we consider several extensions of the polygon triangulation theorem to spaces where the Jordan curve theorem does not hold.

a. Let $P$ be a simple polygon that lies entirely in the interior of the unit square $\square = [0, 1]^2$.
   Prove that the area between $\square$ and $P$ can be triangulated. Every vertex of the triangulation must be a vertex of either $P$ or $\square$.

b. More generally, let $P_0, P_1, P_2, \ldots, P_k$ be pairwise-disjoint simple polygons such that the interior of $P_0$ contains all the other polygons, but otherwise the interiors are disjoint. Prove that the area between $P_0$ and the other polygons $P_i$ (usually called a polygon with holes) can be triangulated using only line segments between the vertices of the various polygons.

c. A spherical polygon is a circular sequence of points connected by great-circle arcs on the sphere. A spherical polygon is simple if it does not self-intersect. The Jordan curve theorem implies that any spherical polygon $P$ divides the sphere into two components, both of which are bounded. Prove that it is possible to triangulate both of these components using great-circle arcs between vertices of $P$.

d. The infinite cylinder is the product $S^1 \times \mathbb{R}$ of the circle and a line. Prove that any geodesic polygon on the infinite cylinder can be triangulated.

e. Prove that any geodesic polygon on the projective plane $S^2/\sim$ can be triangulated.

f. Prove that any geodesic polygon on the flat square torus $S^1 \times S^1$ can be triangulated.
2. Compatible triangulations

For any simple polygons $P$ and $Q$, the Dehn-Schönflies theorem implies that there is a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(P) = Q$. Moreover, if $P$ has $n$ vertices $p_1, p_2, \ldots, p_n$ and $Q$ has $n$ vertices $q_1, q_2, \ldots, q_n$, we can further require that $\phi(p_i) = q_i$ for every index $i$. This exercise asks you to construct such a homeomorphism explicitly.

Let $\square$ be a square that is large enough to comfortably contain both $P$ and $Q$. We say that a triangulation $T$ of $\square$ supports $P$ if every vertex of $P$ is a vertex of $T$ and every edge of $P$ is the union of edges of $T$. (Unlike the previous problem, vertices of $T$ are not required to be vertices of $P$ or $\square$.) Two triangulations $T_P$ and $T_Q$ of $\square$ with labeled vertices are compatible with $P$ and $Q$ if they satisfy the following conditions:

- $T_P$ and $T_Q$ are isomorphic as labeled planar maps. That is, the vertex labeling induces bijections between the vertices, edges, and faces of $T_P$ and the vertices, edges, and faces of $T_Q$, respectively.
- Corresponding vertices on the boundary of $\square$ have the same coordinates in both triangulations.
- $T_P$ supports $P$, and $T_Q$ supports $Q$.
- The vertex labeling also induces bijections between the vertices, edges, and interior faces of $P$ and the vertices, edges, and interior faces of $Q$, respectively. In particular, for any index $i$, vertices $p_i$ and $q_i$ have the same label in $T_P$ and $T_Q$, respectively.

![Figure 1: Compatible labeled triangulations of two simple polygons](image)

a. Describe an algorithm to compute compatible triangulations for two given $n$-gons with at most $O(n^2)$ vertices. (This implies a piecewise-linear homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with complexity at most $O(n^2)$ that is the identity outside the bounding box $\square$.)
b. Prove that the $O(n^2)$ upper bound cannot be improved in the worst case.
c. (?) Describe an efficient algorithm to determine if two simple $n$-gons have compatible triangulations with exactly $n + 4$ vertices, namely, the vertices of the polygon plus the vertices of the bounding box $\square$.
d. (!!) Prove that computing compatible triangulations with the minimum number of vertices is NP-hard.
Describe an algorithm that either computes triangulations of the flat square torus that are compatible with two given geodesic polygons $P$ and $Q$, or correctly reports that no such triangulation exists. (A compatible triangulation exists if and only if $P$ and $Q$ are either both contractible or both essential.) How many vertices are required in the worst case?

It may be easier to start by considering compatible triangulations only of the interiors of the polygons, as described by Aronov, Seidel, and Souvaine [CGTA 1993]. A similar problem for arbitrary point sets was previously considered by Saalfeld [SOCG 1989].

3. Winding numbers

The lecture notes offer two different definitions of the winding number of a polygon $P$ around a point $o$:

- The sum of the signed angles at $o$ subtended by the edges of $P$
- The number of positive crossings, minus the number of negative crossings, of edges of $P$ by an arbitrary ray leaving $o$.

Prove that these two definitions are in fact equivalent. (Hint: Triangulate $P$.)

4. Fast and Loose with more fingers

Tired of the simple centuries-old game of Fast and Loose that everyone already knows, con artists Tenn and Peller are trying to develop more complex variants.

a. In their first variant, they plan to place the chain on the table so that it forms three loops, and then invite the mark to put fingers into two of them. The mark wins if the chain is held fast to the table by their two fingers. Placing fingers in all three loops must hold the chain fast to the table. Describe how Tenn and Peller can always win. Equivalently, describe a closed curve $C$ in the plane and three points $p, q, r$ such that $C$ is contractible in $\mathbb{R}^2 \setminus \{p, q\}$ and in $\mathbb{R}^2 \setminus \{p, r\}$ and in $\mathbb{R}^2 \setminus \{q, r\}$, but not contractible in $\mathbb{R}^2 \setminus \{p, q, r\}$. How long is the crossing sequence of your curve?

b. In the harder variant, they place the chain so that it forms $n$ loops, and then invite a crowd of marks to place fingers into any $n - 1$ of them. Equivalently, they want to want to design a curve that is non-contractible in the plane minus $n$ points, but that becomes contractible if we ignore any one of those $n$ points. The crossing sequence of your curve should be a small polynomial in $n$.

See “Picture-Hanging Puzzles” by Demaine et al. [TCS 2013] for a different framing of this problem. For a more formal treatment, see Gartside and Greenwood [Fund. Math. 2007].

5. Polygon homotopy

Fix an arbitrary point $o$ in the plane, called the origin. Let $P$ be a polygon in the punctured plane $\mathbb{R}^2 \setminus \{o\}$ with vertices $p_0, p_1, \ldots, p_{n-1}$. A vertex move on $P$ replaces an arbitrary vertex $p_i$ with a new point $q_i$. This vertex move is safe if it preserves the winding number of the polygon around
the original, that is, neither of the triangles $\triangle p_iq_ip_{i-1}$ and $\triangle p_ip_{i+1}q_i$ contains $o$. (All index arithmetic is modulo $n$.) A sequence of safe vertex moves describes a homotopy between two polygons with the same number of vertices and the same winding number around the origin.

a. Let $n$ be an arbitrary odd integer. Let $P$ be a regular star polygon with $n$ vertices spaced evenly around the unit circle, with winding number $\lfloor n/2 \rfloor$ around the origin, as shown below. Describe how to rotate $P$ around the origin by half a circle using $O(n)$ safe triangle moves.

![Figure 2: Rotating a star polygon](image)

b. Let $P$ and $Q$ be two arbitrary polygons in $\mathbb{R}^2 \setminus \{o\}$ with the same number of vertices $n$ and the same winding number around the origin. Describe how to transform $P$ into $Q$ using $O(n)$ safe vertex moves. [Hint: Aim for a canonical polygon with the correct winding number; the star polygon in part (a) may not be the best candidate.]

c. (!!) Now let $O$ be a finite set of obstacle points in the plane, and let $P$ and $Q$ be homotopic polygons in $\mathbb{R}^2 \setminus O$ with the same number of vertices. How many safe triangle moves are necessary and sufficient to transform $P$ into $Q$ in the worst case? (Even the case $k = 2$ appears to be open.)

d. (!!) Finally, let $P$ and $Q$ be two arbitrary polygons in $\mathbb{R}^2$ with the same number of vertices $n$ and the same rotation number. Now call a vertex move dull if it preserves the rotation number of the polygon; that is, as vertex $p_i$ moves, the angle $\angle p_{i-1}p_ip_{i+1}$ is never zero. (Mnemonically, the corner is never sharp.) Is it always possible to transform $P$ into $Q$ using $O(n)$ dull vertex moves?