Winding numbers + homotopy

Are these two polygons homotopic in $\mathbb{R}^2 \setminus 0$?
Compute $\text{wind}(P,0) = \text{wind}(Q,0)$?

What if there's more than one obstacle?

Topological properties

\begin{align*}
\downarrow \quad \text{Combinatorial objects} \\
\text{Computational}
\end{align*}

Crossing sequence

\[ BAabBAaabB \]

- Same crossing sequence $\Rightarrow$ homotopic
- Equivalence of crossing seqs $\Rightarrow$ Unique minimum equiv. reduction
- Homotopic $\iff$ same reduced crossing sequence

Computed quickly
Fix finite set $D = \{a, b, c, \ldots\}$ with no two on a vertical line.

**Lemma:** Two polygons with same crossing sequence are homotopic.

**Proof:**

Place sentinel points near obstacles.
Reroute crossings thru sentinels.
Move rest of polygon down to a common basepoint.
Elementary reduction
- delete $Aa$ or $aA$ substring
- or $A \rightarrow a$
- or $a \rightarrow A$

Elementary expansion
is reverse of elem. reduction

Two xing seq's are equivalent
if connected by elem.
reductions and expansions

Reduced if no reductions possible

Lemma: Every xing sequence is equivalent to
a unique reduced sequence.

Left greedy reduction

Random
def LeftGreedyReduce(X):
    n = size(X)
    Y = [0 for _ in range(n)]  // reduced sequence = stack
top = -1  // top stack index

    // ----- linear reduction ----- 
    for i in range(X):
        if top < 0 || (X[i] != -Y[top]):  // empty or no match
            top++
            Y[top] = X[i]  // push
        else:
            top--  // pop

    // ----- cyclic reduction ----- 
    bot = 0
    while (bot < top) and (Y[bot] = -Y[top]):
        bot++
        top--

    // ----- done! ----- 
return Y[bot:top+1] 

Proof: Pick cyclic string w

reductons w->x  w->y

Induction
Equivalent \( x \neq y \)

\[ X \leftrightarrow w_1 \leftrightarrow w_2 \leftrightarrow \ldots \leftrightarrow w_k = Y \]

Suppose \( w_{i-1} \leftrightarrow w_i \rightarrow w_{i+1} \)

either \( w_{i-1} = w_{i+1} \) or \( w_{i-1} \rightarrow w_i \leftrightarrow w_{i+1} \)

Induction:

\[ x \rightarrow \rightarrow \rightarrow Z \leftrightarrow \leftrightarrow \leftrightarrow y \]

either \( x \neq Z \) or \( y \neq Z \)

\( x \) is not reduced
or \( y \) is not reduced \( \square \)

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**Theorem:** Two polygons are homotopic in \( \mathbb{R}^2 \setminus \mathbb{O} \) iff they have the same reduced xing sequence.

**Proof:**

\( \Rightarrow \) Every safe vertex move changes xing seq by elem. reductions + expansions

Homotopic polygons have equiv xing sequences

\( \Rightarrow \) equal reduced xing sequences expansion

\( \Leftarrow \) Every elementary reduction can be effected by a sequence of safe vertex moves
Thm: Given two $n$-gons $P$, $Q$ and $k$ points $O$, we can decide if $P$ and $Q$ are homotopic in $\mathbb{R}^2 \setminus O$ in $O(k \log k + kn)$ time.

Proof:
1. Sort $O$ → $O(k \log k)$
2. Compute crossing sequences of $P$ and $Q$ → $O(nk)$ time
   $O(n + k + O \log k)$
3. Reduce crossing seqs $O(x) = O(nk)$
4. Compare reduced sequences $A \equiv B \equiv 3 \equiv 3$ $\Rightarrow$ $O(x) = O(nk)$