**Generic closed curves**

Closed curve \( \gamma: S^1 \to \mathbb{R}^2 \)

Self-intersection \( \gamma(t) = \gamma(t') \)

Transverse: For small enough subcurves \( \gamma(t-e, t+e) \) and \( \gamma(t'-e, t'+e) \), they are homeomorphic to two orthogonal lines.

A curve is **generic** iff

- Every self-int is transverse
- Every self-int is pairwise
- No \( \gamma(t) = \gamma(t') = \gamma(t'') \)

Any curve approximated by a polygon

\[ \text{perturb} \]

Generic polygon

If generic closed curve

# self-intersections is finite

Two curves are isotopic if they are homotopic through generic curves with same # of self-intersections.

Equiv (in \( \mathbb{R}^2 \)) homeomorphism \( H: \mathbb{R}^2 \to \mathbb{R}^2 \)

s.t. \( \gamma = H \circ \gamma' \)
Image graph
vertices = self-ints
edges = sub-curves between vertices

Almost always connected 4-regular plane graph.

Simple \(\Rightarrow\) 0 no vertices!

But not every conn. 4-reg. planar graph is image of a curve

Every curve is an Euler tour of its image

Gaussian Euler tour
After entering any node leave thru opposite edge

image of mult-curve
\(\psi: S^1 \cup S^1 \cup \ldots \cup S^1 \rightarrow \mathbb{R}^2\)

Faces of a curve = Faces of image graph
= components of \(\mathbb{R}^2 \setminus \text{image}\)
= open disks except outer face (comp. of closed disk)

monogon
bigon
triangle
Homotopy moves: Change a small neighborhood of one face with \( \leq 3 \) vertices.

Every homotopy between polygons \( \Rightarrow \) vertex moves \( \Rightarrow \) finite sequence of homotopy moves.

Theorems: Every planar curve with \( n \) vertices has \( n+2 \) faces.

\( \text{(Euler's formula)} \)

\[ V - E + F = 2 \]

\[ E = 2V \]

\[ F \]

Proof: Fix \( Y \)

\[ \gamma_1 \rightarrow \gamma_2 \rightarrow \gamma_3 \rightarrow \cdots \rightarrow \gamma_k = 0 \]

\( F \rightarrow F-1 \rightarrow F-2 \)

\( F-n \) is preserved

\( F=2 \)

\( n=0 \)

2 \( \Rightarrow 0 \) move either merges two faces or disconnects the image graph. \( \uparrow \) impossible
Note added after lecture: This is the correct convention. The vertex signs and Gauss code above right are also correct.

Note added after lecture: These instructions are for tracing a face counterclockwise (on the left of a moving point), but the two examples in the Gauss diagrams are tracing the face clockwise (on the right).

Signed Gauss code is consistent with a planar curve $F = n + 2$.