Minimum Cuts

\[ \Sigma = \text{planar map} \]

\[ s, t = \text{vertices} \]

every edge has a capacity \( c(e) \geq 0 \)

\((s,t)-\text{cut} = \text{subset of edges intersecting every path } s \to t\)

\( \Sigma \) is undirected

e.g. image segmentation

mesh/model segmentation

\[ \Sigma^* \] is homotopic to \( \partial s^* \) in \( \Sigma \)

\( \Gamma = \text{shortest path from } s^* \text{ to } t^* \)

\( \gamma^* = \text{shortest essential cycle (unique)} \)

\( \Gamma \) and \( \gamma^* \) cross exactly once

Lemma:
q→p_i enters from left

\[ = p_{i-1} \rightarrow p_i \quad q \rightarrow p_i \quad p_{i+1} \rightarrow p_i \]

in clockwise order around \( p_i \)

negative crossing a subpath of \( \mathcal{Y} \)
enters \( \Pi \) on left
leaves \( \Pi \) on right
stays on \( \Pi \) otherwise

Proof:
Suppose essential cycle \( \mathcal{Y} \) that crosses \( \Pi \) more than once

Orient \( \mathcal{Y} \) so that wind = +1

Somewhere \( \mathcal{Y} \) has neg crossing
followed by pos crossing
\( \Rightarrow \) \( \mathcal{Y} \) has subpath
leaves \( \Pi \) to right @ \( p_i \)
enters \( \Pi \) from right @ \( p_{i+1} \)

Replace with \( \Pi (p_i, p_{i+1}) \)

makes \( \mathcal{Y} \) shorter
so \( \mathcal{Y} \) not shortest ess. cycle \( \Pi \)

Slice \( \Sigma^* \) along \( \Pi \)
shortest essential cycle in $\mathbb{Z}^*$

shortest path from $p_i^-$ to $p_i^+$ in $\mathbb{Z}^*/\pi$

over all $p_i^-$

[Itai-Shiloach '79]

\[ \operatorname{Dijkstra}: \quad O(kn \log n) \quad k = \# \pi \]

Faster:

\[ O(kn) = O(n^2) \]

for all $p_i^-\pi^-\pi^+$

\[
\text{find dist} \quad (p_i^+, p_i^-) \quad \text{min.} \\
\]

\[ O(n \log n) \]

MSSP!

\[ \begin{align*}
\operatorname{Divide \& conquer} & \quad O(n \log n) \\
\text{[Klein '05]} & \quad O(n \log k) \\
\text{[Reif '83]} & \quad \left[ \text{[Frederickson '87]} \right] \\
\end{align*} \]

\[ O(n \log n \log k) \]

\[ \begin{align*}
\text{compute median path } O(n) \\
\text{from } P_i^- \text{ to } P_i^+ \\
\text{recursion on both sides} \\
\text{in both components of } \quad \Delta \quad \text{or } \quad \omega \\
\end{align*} \]

\[ T(n, k) = T(n_1, k/2) \cdot T(n_2, k/2) + O(n) \]

where $n_1 + n_2 = n$

\[ \Rightarrow O(n \log k) \]
IF floor+ceiling share vertex x
Compute shortest paths from x
\[ \text{dist}(p_i^*, p_i) = \text{dist}(r, p_i^*) + \text{dist}(x, p_i) \]

But we can do better \([O(n \log \log n)]\) sep Monge
Last lecture Bellman-Ford \([O(n^2)]\)
Next lecture Dijkstra \(\Rightarrow \text{FR-Dijkstra}\)

Build DDG \([O(n \log r)]\)
Dijkstra in DDG
\[ O(E + V \log V) = O(n + \frac{n}{r} \log r) \]

\[ \frac{n}{r} \log r \]
Underlying data structure: Monge heap online SMAWK
bdry to bdry distances in $O(n \log n + k^2 \log n)$

bdry = nice separator $O(n \log n)$

Open = $O(n \log \log n)$? $O(n)$?