Planar section

Find topologically interesting cycles in surface maps.

non-contractible = non-trivial homotopy,
non-separating = non-trivial homology

Thomassen 90] 3-path condition

\[\begin{align*}
\alpha \cdot \beta & \geq \\
\beta \cdot \gamma & \geq \\
\alpha \cdot \gamma & \geq 
\end{align*}\]

If two of these are trivial so is the third.

⇒ Assume edges of \(\Sigma\) have non-neg. weights

\(\sigma\) = shortest non-trivial cycle

\(x, y = \text{antipodal pts on } \sigma\)

\(a\) and \(Y\) are shortest paths in \(\Sigma\).

Proof: Suppose not. Let \(\beta\) be true shortest path from \(x\) to \(y\).

\(\alpha \cdot \beta\) is shorter than \(\sigma\) ⇒ trivial

\(\beta \cdot \gamma\) is shorter than \(\sigma\) ⇒ trivial

3-path ⇒ \(\sigma\) is trivial
For all vertices $x$, shortest path tree $T_x$ for all edges $e \in T_x \leq O(n)$

examine loop $(T_x, e) \leq O(n)$ time

$O(n^3)$ time

Greedy tree-root tree decomposition $(T_i, L_i, C_i)$

$T = \text{shortest path tree rooted at } a \cdot b$-vertex $x$.

$C = \max. \text{spanning tree of } \Sigma *

w(e*) = \text{length of loop } (T, e)

$L = E \setminus (T \cup C)$

Greedy system of loops: $L = \bigcup \text{loop}(T_i, e) \mid e \in L_i$

Any system of loops $L$ is non-separating

- Every loop based at $x$ is homotopic to sequence of loops in $L$

  "basis of $T_x$"

- Slicing $\Sigma$ along $L$ leaves a disk.

Greedy:

- $L$ is the shortest system of loops based at $x$

[EW '05, CDV '10]

One-casing condition

The shortest nontrivial cycle $\sigma$ crosses any shortest path $\pi$ at most once.

Proof: same exchange arg.
Dual cut graph $K^* = C^* \cup L^*$ subgraph of $E^*$ with one face
$= E^* \setminus K^*$ is a disk

Greedy Reduced dual cut graph $\Gamma Z^*$
repeatedly remove degree 1 vertices ("hair") from $K^*$

Lemma: loop($T$, $e$) or cycle($T$, $e$)
is separating
$\iff K^* \setminus e^*$ is disconnected
$\iff e^*$ is a bridge of $K^*$
$- e^*$ is "hair" $\implies$ contractible loop
$- e^*$ is bridge in $R^* \implies$ noncontractible

For any basepoint $x$ find shortest nontrivial loop based at $x$ in $O(n \log n)$ time

$O(n \log n) \rightarrow$ Dijkstra $\rightarrow$ $T$
$O(n \log n) \rightarrow$ Borůvka $\rightarrow$ $C^* \cup L^*$
$\rightarrow$ bookkeeping $\rightarrow$ $K^*$
$\rightarrow$ $T$ FS $\rightarrow$ $\Gamma Z^*$
$O(n)$
$\rightarrow$ min wrt edge $e^* \in R^*$ $\rightarrow$ shortest non-con loop
$\rightarrow$ min wrt non-bridge $e^* \in R^*$ $\rightarrow$ shortest non-sep loop
$O(n) \rightarrow O(n^2)$ find shortest overall $[EH 03]$ $[Cd 10]$

Don't try every basepoint at a time
CC ID

AG

EF

Erg

aecomp

cycle

CT
es
I
ee
L
zonmone
sq
enFIE
ieasEonee

Shortest non-sep cycle crosses
some cycle in \( \Pi \) at least once

Every non-sep cycle crosses
some cycle in \( \Pi \) at least once

Shortest non-sep cycle crosses
each cycle in \( \Pi \) at most once

For each \( ee \in \mathcal{L} \)
slice \( \geq 1 \) along cycle \( CT(e) \)

MSSP \( \leftrightarrow \) shortest cycle
crossing (cycle\(CT(e)\)) once

MSSP for surfaces

Tree-cotree \( \rightarrow \) Tree-grove dynamic forest

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Each dart enters \( \Gamma \) at most \( 2g + 1 \) times

\( O(G \log n) \) time per pivot

\( O(G^2 n \log n) \) time overall

\( O(g^2 n \log n) \) time

Conjecture: \( O(g \cdot n \log n) \)