Minimum cuts in surface maps — Homology

Cycle space

- Space of all subgraphs of \( G \) generated by cycles under sym diff \( \oplus \)
- Elements are even subgraphs
- Vector space \( \mathbb{Z}_2^{E-V+1} \)
- Generated by fundamental cycles wrt any spanning tree

Cut space

- Space of subgraphs that cross partitions of \( V \)
- Elements are edge cuts
- Vector space \( \mathbb{Z}_2^{V-1} \)
- Generated by fundamental cuts wrt any spanning tree

Cycle space \( \mathbb{Z}_2^E \) — space of all subgraphs
- \( V \) linear constraints at vertices
- But 1 redundant

Boundary space

- Boundary of subset \( A \subseteq F \)
  - All edges with one shore in \( A \)
- All boundary subgraphs
  - \( \mathbb{Z}_2^{F-1} \)
- Every boundary subgraph is even.
- Not every even subgraph is boundary!

Planar maps:

\[
\text{Cycle}(\Xi) = \text{Cut}(\Xi^*) \quad \mathbb{Z}_2^{F-1}
\]
Two subgraphs of surface map $\Sigma$ are homologous iff $A \oplus B$ is a boundary.

First homology group / Homology space:

\[ H_1 = \text{cycles} / \text{boundaries} \]

set of homology classes of even subgraphs

\[ A \sim A', \quad B \sim B' \implies A \oplus B \sim A' \oplus B' \]

\[ H_1 = \mathbb{Z}_2^{n-v+1} / \mathbb{Z}_2^{n-1} = \mathbb{Z}_2^{g} \]

Testing homology

Given subgraph $A$ of $\Sigma$, is $A$ a boundary?

$O(n)$ time dual WFS

Computing homology

Tree-cotree decomposition $(T, L, C)$

system of cycles: $\Gamma = \sum_{i=1}^{g} \xi_i$, $\delta_i = \text{cycle}(T, e_i)$, $e_i \in L$

basis for $H_1$

Every even subgraph is hom. to sum of cycles in $\Gamma$:

system of cocycles: $\Lambda = \sum_{i=1}^{g} \lambda_i$

\[ \lambda_i = \text{cycle}(C^i, e_i^*) \quad e_i \in L \]
For every edge \( e \) of \( \Sigma \) define \([e] \in \mathbb{Z}_2^{\Sigma}\) for every \( i \): \([e]_i = [e \in \lambda_i^i]\)

Homology class of \( A = \bigoplus_{e \in A} [e] = [A] \in \mathbb{Z}_2^{\Sigma}\).

Poincaré duality:

Min cut in planar map \( \Sigma \):

- Shortest cycle in \( \Sigma^* \backslash \{s^*, t^*\}\)

homologous with \( \partial s^* \)

Dual:

Even subgraph with min cut homologous with \( \partial s^* \)
in \( \Sigma^* \backslash \{s^*, t^*\}\)

Even subgraph homol. with \( \gamma \) in \( \Sigma^* + \text{handle} \)

Find shortest cycle in given homology class, \( h \in \mathbb{Z}_2^{\Sigma}\)

\( \Sigma = \mathbb{Z}_2 \)-homology cover of \( \Sigma \)

\( = (\emptyset, \hat{e}, \hat{F}) \)

Ghe \( 2^{2g} \) copies of \( \Sigma \) together complexity \( 2^{2g,n} \), genus \( 2^{2g} \).
\[ \Phi = \{ (v, h) \mid v \in V, \ h \in \mathbb{Z}_e^{2g} \} \]
\[ \hat{E} = \{ (u, h)(v, h') \mid u \sim v, \ h \oplus h' = \Sigma uvf \} \]

\[ \text{shortest cycle in } \hat{E} \text{ contains } v \text{ in homology class } h \]
\[ \text{shortest path in } \hat{E} \text{ from } V_{000...0} \text{ to } V_n \]

Try all \( v \) and compute shortest path from \( u \) to \( v \) in \( \tilde{E} \) \( n \log n \) steps.

\[ \tilde{O}(p_0(n \log n)) = 2^{O(g)} n \log n \]

\[ \text{MSSP} \Rightarrow \text{shortest cycle in any hom. class} \quad 2^{O(g)} n \log n \]

\[ \min \text{ wt even subgraph} \quad 2^{O(g)} n \log n \]

\[ \text{NP-hard} \]

\[ \mathcal{O}(g^s \cdot n) ? \]