DISCRETE AND COMPUTATIONAL GEOMETRY:

TEN YEARS LATER

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Approximating Shortest Paths on a Convex Polytope in Three Dimensions

Pankaj K. Agarwal

Given a convex polytope P with n faces in $\Re^3$, points $s, t \in \partial P$, and a parameter $0 < \varepsilon \leq 1$, we present an algorithm that constructs a path on ∂P from s to t whose length is at most $(1+\varepsilon)d_P(s,t)$, where $d_P(s,t)$ is the length of the shortest path between s and t on ∂P. The algorithm runs in $O(n \cdot \min\{1/\varepsilon^{1.5}, \log n\} + 1/\varepsilon^3)$ time, and is relatively simple to implement. We also present an extension of the algorithm that computes approximate shortest paths from a given source point on ∂P to all vertices of P.

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Video Presentation

Four-polytopes and a funeral (for my conjecture)*

Nina Amenta† **Tamara Munzner**‡

A four-dimensional polytope given as the intersection of f halfspaces has $O(f^2)$ vertices. The (incorrect) conjecture of the title is that $o(f^2)$ of these vertices can appear on the outer boundary of a projection to the plane (a "shadow").

The video illustrates the construction of a four-polytope with an $\Omega(f^2)$ shadow [AZ]. It uses an idea, due to Klee and Minty [KlMi] and applied by others, for constructing polytopes with long simplex paths.

The projection of the polytope to R^3 is shown, as it rotates in R^4, so that the four-dimensional object is understood as a continuous family of deformations of a three-dimensional object. Structure is shown by drawing only subsets of faces, related by the edge skeleton which is always shown.

The video appears as part of an annual collection of computational geometry videos [Am]. It was made at The Geometry Center on an SGI IRIS using Geomview [LMP] and the StageManager scripting system [Ce] . The polytope models were constructed using the convex hull program chD [Em].

References

[AZ] Nina Amenta and Günter M. Ziegler. Deformed Products and Maximal Shadows, submitted to *Discrete and Computational Geometry: Ten Years Later*, AMS Series in Contemporary Mathematics. *See also*
Nina Amenta and Günter M. Ziegler. Shadows and slices of polytopes, *Proceedings of the 12th Annual ACM Symposium on Computational Geometry* (1996), pages 10-19.

[Am] *5th Annual Video Review of Computational Geometry*, Nina Amenta, ed. AMS (1996).

[Ce] Davide Cervone. `Stagemanager`, not yet available.

[Em] Ioannis Emiris. `chD`, `ftp://robotics.eecs.Berkeley.edu/pub/ConvexHull`.

[KlMi] Victor Klee and George J. Minty. How good is the simplex algorithm?, *Inequalities III*, (O. Shisha, ed.), Academic Press, New York, (1972), pages 159–175.

[LMP] Stuart Levy, Tamara Munzner and Mark Phillips. `Geomview`, `ftp://geom.umn.edu/pub/software/geomview`.

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Visibility with Multiple Reflections

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Abstract

We show that the region lit by a point light source inside a simple n-gon after at most k reflections off the boundary has combinatorial complexity $O(n^{2k})$, for any $k \geq 1$. A lower bound of $\Omega((n/k)^{2k})$ is also established which matches the upper bound for any fixed k. A simple near-optimal algorithm for computing the illuminated region is presented, which runs in $O(n^{2k} \log n)$ time and $O(n^{2k})$ space for any $k > 1$.

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Affine perimeter and limit shape

Imre Bárány
Mathematical Institute, Budapest

We prove here that, given a convex compact set $K \subset R^2$, almost all convex $\frac{1}{n}Z^2$–lattice polygons contained in K are very close to a fixed convex set $K_0 \subset K$ as n goes to infinity. The distinguishing property of K_0 is that it has the largest affine perimeter among all convex sets contained in K. The methods can be used to determine the asymptotic behaviour of the probability that k random, independent points drawn uniformly from K form the vertices of a convex k–gon. Actually, there is a limit shape to these convex random k–gons as k increases and this limit shape is the same set K_0.

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Computing mixed discriminants, mixed volumes and permanents

Alexander Barvinok

We present a probabilistic polynomial time algorithm that computes the permanent of any given non-negative n by n matrix within a factor $2^{O(n)}$. That is, for any given non-negative matrix A the algorithm computes a number α which satisfies the inequalities c^n per $A \leq \alpha \leq$ per A with the probability at least 0.9, where $c > 0$ is an absolute constant (one can choose $c = 0.28$). The algorithm solves a more general problem of approximating the mixed discriminant of given n positive definite operators on $\mathbf{R}^n$. Similarly, the mixed volume of n given ellipsoids in $\mathbf{R}^n$ can be approximated in polynomial time within a factor c^n for $c = 0.66$. The algorithms are based on a recursive application of the projective version of the kinematic formula which allows us to represent the mixed discriminant (mixed volume) in $\mathbf{R}^n$ as the expectation of the mixed discriminant (mixed volume) in a random hyperplane in $\mathbf{R}^{n-1}$. The paper is available at http://www.math.lsa.umich.edu/~barvinok/papers.html

Bounding the topological complexity of semi-algebraic sets

Saugata Basu

In this talk I will describe a recent result on bounding the sum of the Betti numbers of semi-algebraic sets. This extends a well-known bound due to Oleinik and Petrovsky , Thom and Milnor. In separate papers they proved that the sum of the Betti numbers of a semi-algebraic set $S \subset R^k$, defined by $P_1 \geq 0, \ldots, P_s \geq 0, deg(P_i) \leq d, 1 \leq i \leq s$, is bounded by $(O(sd))^k$. Given a closed semi-algebraic set $S \subset R^k$ defined as the intersection of a real variety, $Q = 0, deg(Q) \leq d$, whose real dimension is k', with a set defined by a quantifier-free Boolean formula with no negations with atoms of the form, $P_i = 0, P_i > 0, P_i < 0, deg(P_i) \leq d, 1 \leq i \leq s$, we prove $P_i = 0, P_i \geq 0, P_i \leq 0, deg(P_i) \leq d, 1 \leq i \leq s$, we prove that the sum of the Betti numbers of S is bounded by $s^{k'}(O(d))^k$. In the special case, when S is defined by $Q = 0, P_1 > 0, \ldots, P_s > 0$, we have a slightly tighter bound of $\binom{s}{k'}(O(d))^k$. This result generalises the Oleinik-Petrovsky-Thom-Milnor bound in two directions. Firstly, our bound applies to arbitrary unions of basic semi-algebraic sets, not just for basic semi-algebraic sets. Secondly, the combinatorial part (the part depending on s) in our bound, depends on the dimension of the variety rather than that of the ambient space.

We also extend this bound to the case where the algebraic complexity is bounded only in terms of the number of monomials appearing in the system of polynomials (the degrees being unbounded). We also note that these bounds are not obtainable by the Thom-Milnor technique and that a new idea of separating the combinatorial part of the complexity from the algebraic part plays a crucial role.

ISOPERIMETRIC INEQUALITIES AND THE DODECAHEDRAL CONJECTURE REVISITED

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The cd-index of zonotopes and arrangements

Louis J. Billera, Richard Ehrenborg and Margaret Readdy

We investigate a special class of polytopes, the zonotopes, and show that their flag f-vectors satisfy only the affine relations fulfilled by flag f-vectors of *all* polytopes. In addition, we determine the lattice spanned by flag f-vectors of zonotopes. By duality, these results apply as well to the flag f-vectors of central arrangements of hyperplanes.

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FIXING AND HINDERING SYSTEMS FOR CONVEX BODIES
Vladimir Boltyanski

Let $M \subset R^n$ be a compact, convex body and $F \subset \operatorname{bd} M$. A vector $v \neq 0$ *remuves the interior of M from F* if $(\lambda v + \operatorname{int} M) \cap F = \emptyset$ for any $\lambda > 0$. The set $F \subset \operatorname{bd} M$ is a *fixing system* [8] for M if there is no nonzero vector v which removes $\operatorname{int} M$ from F. A fixing system F is *primitive* if no proper subsystem $F' \subset F$ is a fixing system for M. Evidently, a fixing system of the least cardinality is primitive. Denote the minimal cardinality of fixing systems for M by $\varrho(M)$. B. Grünbaum established [7] that for any compact, convex body $M \subset R^n$ the estimate $n + 1 \leq \varrho(M) \leq 2n$ holds.

A vector $v \neq 0$ *remuves the body M from F* if $(\lambda v + M) \cap F = \emptyset$ for any $\lambda > 0$. The set $F \subset \operatorname{bd} M$ is a *hindering system* [9] for the body M if there is no nonzero vector v which removes M from F.

Finally, let $v(t)$, $0 \leq t \leq 1$, be a continuous family of vectors in R^n such that $v(0) = 0$. We say that the family $v(t)$ *remuves the body M from F* if for $0 < t \leq 1$ the equality $(v(t) + M) \cap F = \emptyset$ holds. The set $F \subset \operatorname{bd} M$ is a *strict hindering system* [3] for the body M if there is no continuous family $v(t)$ with $v(0) = 0$ which removes M from F.

In the talk, the following new results by H. Martini, E. Morales and the speaker [3, 4, 6] are formulated (the estimates hold for any compact, convex body $M \subset R^n$; the functional $\operatorname{md} M$ is introduced in [2]):

$$2 \leq \sigma(M) \leq \sigma'(M) \leq \operatorname{md} M + 1 \leq n + 1, \qquad ([3], 1996)$$

$$n + 1 \leq n + \frac{n}{\operatorname{md} M} \leq \varrho(M) \leq 2n + 1 - \operatorname{md} M \leq 2n. \qquad ([4, 6], 1995)$$

For *maximal* cardinalities of primitive fixing and hindering systems of planar, compact, convex figures, the complete analysis conducted by L. Fejes Tóth [8] and P. Many [9].

In the case of the bodies $M \subset R^n$ for $n \geq 3$ the situation is in a sense unexpected. Denote by $\varrho_{\max}(M)$, $\sigma_{\max}(M)$, $\sigma'_{\max}(M)$ the maximal cardinalities of primitive fixing, hindering, and strict hindering systems for the body M, respectively. As B. Bollobás [1] showed, for any integer $k \geq 4$ there exists a body $M_k \subset R^3$ with $\varrho_{\max}(M_k) \geq k$. In our joint paper with H. Martini [5], this result is improved. Namely, consider in R^3 the body $P = \operatorname{conv}(B \cup \{a\})$, where B is a ball and $a \notin B$. Then $\varrho_{\max}(P) = \infty$, i.e., for any integer $k \geq 4$ there is a primitive fixing system $F_k \subset \operatorname{bd} P$ that consists of k points. The same holds for hindering and strict hindering systems [3].

1. Bollobás B. Fixing system for convex bodies. Studia Sci. Math. Hungar. 2 (1967), 351-354.
2. Boltyanski V. Helly's theorem for H-convex sets. Soviet Math. Dokl. 17 (1976), no. 1, 78-81.
3. Boltyanski V. On hindering systems for convex bodies (to appear in Mat. Sbornik, 1996).
4. Boltyanski V. and Martini H. Combinatorial geometry of belt bodies. Results Math. 28 (1995), 224-249.
5. Boltyanski V. and Martini H. On maximal primitive fixing systems. Beiträge zur Algebra und Geometrie, 37 (1996).
6. Boltyanski V. and Morales E. Minimal fixing systems for convex bodies. Jornal of Applied Analysis 1 (1995), no. 1, 1-12.
7. Grünbaum B. Fixing systems and inner illumination. Acta Math. Acad. Sci. Hungar. 15 (1964), 161-168.
8. Fejes Tóth L. On primitive polyhedra. Acta Math. Acad. Sci. Hungar. 13 (1962), 379-383.
9. Mani P. On polytopes fixed by their vertices. Acta Math. Acad. Sci. Hungar. 22 (1971), 269-273.

On the difficulty of vertex and facet enumeration

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A *convex polytope* P can be specified in two ways: as the convex hull of the vertex set $\mathcal{V}$ of P, or as the intersection of the set $\mathcal{H}$ of its facet-inducing halfspaces. The *vertex enumeration problem* is to compute $\mathcal{V}$ from $\mathcal{H}$. The *facet enumeration problem* it to compute $\mathcal{H}$ from $\mathcal{V}$. These two problems are essentially equivalent under point/hyperplane duality. Although there are efficient algorithms for non-degenerate input, it is open whether vertex/facet enumeration for arbitrary input can be solved in time polynomial in $|\mathcal{H}| + |\mathcal{V}|$.

Known methods for facet enumeration (and in the dual setting, vertex enumeration) fall into one or more of the the following categories:

1. lattice producing algorithms compute the entire face lattice,
2. triangulation producing algorithms compute a triangulation of the boundary complex (possibly by perturbation), and
3. incremental algorithms compute the final output by inserting the input points one at a time and maintaining a series of intermediate polytopes.

Each of these methods has a weakness: the face lattice, triangulation, and intermediate polytopes can all have size superpolynomial in $|\mathcal{H}| + |\mathcal{V}|$. In this talk we present a number of families of polytopes hard for one or more of these methods and several families that are hard for all three.

This talk summarizes two recent papers: "How good are convex hull algorithms?" by Avis, Bremner, and Seidel, will appear in *Computational Geometry: Theory and Applications*; "Incremental convex hull algorithms are not output sensitive." by Bremner is currently in preparation.

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A polynomial time approximation technique for Norm-Maximization

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Let $K \subseteq \mathbb{R}^d$ be a convex body (i.e. a compact, convex and full-dimensional set) and define for $x = (x_1, .., x_d)^T \in \mathbb{R}^d$ and $p \in \mathbb{N}$, $\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}}$. Then the problem of computing the p^{th} power of $\max_{x\in K} \|x\|_p$ is $\mathbb{NP}$-hard even if K is a parallelotope centered at the origin (*Bodlaender, Gritzmann, Klee* and *van Leeuwen*).

In the realm of the algorithmic theory of convex bodies developed by *Grötschel, Lovász* and *Schrijver*, it can be inferred from work by *Bárány* and *Füredi* that for p=2 it is not possible to compute in (oracle-) polynomial time the maximum within an error of $\gamma_1\sqrt{d/\log d}$, γ_1 a constant independent on d, for convex d-dimensional bodies given by separation oracles.

Inspired by work of *Kochol* we use a 'good' spherical code to show that $\max_{x\in K} \|x\|_2$ can indeed be approximated within an error less than $\gamma_2\sqrt{d/\log d}$ (γ_2 a constant independent on d).

A similar technique can also be applied in general l_p-spaces. The result can further be used to give optimal approximative algorithms for certain inner and outer radii of convex bodies, particularly for inradius.

*Joint work with Peter Gritzmann, Trier, and Victor Klee, Seattle

Computational Geometry: The Next 10 Years

Bernard Chazelle

The last 10 years have been the most exciting in the history of computational geometry. Stunning progress has been made in many areas: the asymptotic complexity of basic geometric computations such as convex hulls, Voronoi diagrams, polygon triangulation, motion planning, low-dimensional optimization, has been resolved. Randomization and a new theory of geometric sampling have produced simplifications and improvements for a large array of geometric algorithms. Computational topology and computational algebraic geometry, as well as the theory of line and surface arrangements have reached a very high level of sophistication.

Less successful has been the application of these theoretical advances to the practice of geometric computing. We will argue that the emphasis on foundations and theoretical investigations was the right course to follow, but that the next decade should see a broadening of the computational-geometric base to encompass problems of practical as well as theoretical relevance.

Some Exotic Species of Aperiodic Tilings in Two and Three Dimensions

Ludwig Danzer

First some basic notions are introduced:
The *protoset* $\mathcal{T} := \{T_1, T_2, \ldots, T_k\}$ (T_i are the *prototiles*);
the *species* $S(\mathcal{T}, \mathrm{cond})$, the family of all global tilings P, such that every tile of P is congruent to some prototile, and P satisfies some condition cond (e.g.: inflation, local matching rule). A species S is said to have property
(I) if S can be defined by an *inflation* infl (the inflation factor is denoted by η),
(D) if (I) and $\mathrm{defl} := \mathrm{infl}^{-1}$ is unique,
(PV) if (I) and η^d is a PV-number,
(LMR) if S can be defined by a local matching rule lmr.

The following example are discussed:
1) AMMAN'S chair. (I), (D), (PV), (LMR), $\eta^2 = \tau$.
2) A species S with (I), (D), (PV) ($\eta^2 = 2$) but a singular inflation matrix (not (LMR)).
3) A species S with three prototiles ($3 = \frac{1}{2}\phi(7)$) each a triangle with all angles of type $k\pi/7$ satisfying (I), (D) and (LMR) (but not (PV)) and η of degree three.
4) BÖRÖCZKY's example in the hyperbolic plane.
5) SCD, a biprism that tiles $\mathbb{E}^3$, but $S(\{\mathrm{SCD}\}, \text{—})$ is aperiodic (cf. problem 1 of the problem session).
6) $S(\{P\}, \mathrm{infl})$, where P is a triangular prism and infl is such that in every tiling belonging to S the copies of P occur in infinitely many orientations, which form a dense subset of $O^+(\mathbb{R}, 3)$ (due to J. H. CONWAY and Ch. RADIN).

Intersection Graphs of Jordan Arcs

H. de Fraysseix and P. Ossona de Mendez *

Abstract

An *intersecting family* of Jordan arcs is a family of Jordan arcs, such that two arcs are neither sharing more than one point nor tangent, and such that one crossing point belongs to exactly two arcs. Such a family defines an *intersection graph*, whose vertices are the arcs, and whose edges are the crossing points. Deciding wether a given graph is the intersection graph of some intersecting family of Jordan arcs is known to belong to the NP-complete class. It has been a challenge for long to prove or disprove that all planar graphs have this property.

Up to now, only those planar graphs representable by contacts (e.g. bipartite and outerplanar graphs) are known to have this property. We extend here this result to 4-connected maximal planar graphs which are either 3-colorable or free of C_4 separating cycle.

Leading Strand

Arc intersection problems arise topological difficulties that vanish when arcs are only in contact.

Recall that a *contact family* of Jordan arcs is a family of Jordan arcs, such that two arcs share at most one point, and such that any point is interior to at most one arc. Besides an intersection graph, called *contact graph* in this context, such a family defines a planar bipartite *point-arc incidence graph*, whose vertices are the arcs and the contact points, and whose edges are defined by inclusion of a contact point into an arc.

We fully characterize, by a pure combinatorial property, those bipartite planar graphs which are point-arc incidence graphs.

To a 4-connected maximal planar graph, either 3-colorable or free of C_4 separating cycles, we associate a bipartite planar graph. Then the result follows from the proof that this graph is a point-arc incidence graph : from the associated contact family, local deformations around the contacts give rise to an intersection family which represents the original graph.

Whenever the graph is not 3-colorable, the proof relies heavily on the four color theorem.

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Computing combinatorial models of real smooth hypersurfaces

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Hilbert's 16th problem concerns the classification of topological types of smooth real projective hypersurfaces. An important component on the classification efforts is to have effective methods to produce examples of new topological types. During the 1980's Oleg Viro and his students developed a technique, based on convex triangulations of Newton polytopes, to construct examples. In this talk we present partial results about Viro's construction when it is extended to arbitrary triangulations. The guiding question is whether Viro's construction still produces combinatorial models of real hypersurfaces when the triangulations of the Newton polytopes are not convex (also called regular). We have proved some results hold in dimension two, such as Harnacks inequality and some consequences of Bezout's theorem. We have written software that allowed us to compute several millions of examples that indicate Petrowsky and Arnol'd inequalities could be valid as well for non-convex triangulations. This is joint work with Frederick Wicklin.

Computational Topology

Herbert Edelsbrunner

The speaker believes there is or should be a reasearch area appropriately referred to as 'computational topology'. A more accurate but also more cumbersome title would emphasize the intended relation to discrete and geometric methods. Its agenda includes the identification and formalization of topological questions in computer applications and the study of algorithms for topological problems. It is hoped this talk can contribute to the creation of a computational branch of topology with a unifying influence on computing and computer applications.

The talk illustrates the interaction between application problems, geometric models, and topological methods with four case studies:

1. Molecular conformation; Voronoi complexes, alpha shapes; holes, homology, filtrations.

2. Imperfect holes; Delaunay complexes, pockets; vector fields, stable manifolds.

3. Molecular surfaces; Pedoe vector space, skin; topological duality.

4. Deformation; Minkowski sum, convexification principle; preimage theorem, Morse theory.

Recent Progress on Packing and Covering

Gábor Fejes Tóth

In the talk we give a survey about new developments in the theory of packing and covering. We start with an account about recent plans for proving Kepler's conjecture concerning the densest packing of congruent balls in three-dimensional Euclidean space. We continue with the discussion of bounds for the packing and covering densities of a convex body. The most important new result on this topic is Rush's construction of dense lattice packings via codes for a special metric. For certain bodies his construction yields considerable improvement upon the Minkowski-Hlawka bound. We also address the question about the regularity of optimal arrangements. Roger's conjecture that for sufficiently large d the densest packing of equal balls in E^d cannot be lattice like, is still open. On the other hand, A. Bezdek and W. Kuperberg showed that for $d \geq 3$, there is an ellipsoid E in E^d such that the densest packing of congruent copies of E cannot be realized in a lattice arrangement. We mention some further results indicating that, in general, optimal arrangements are not very regular.

Algorithms for prediction of indoor radio propagation

Steven Fortune
Bell Laboratories

The design of indoor wireless communication networks requires prediction of radio propagation inside a building. Radio propagation inside a building is complex, because each building wall can act both as an obstruction, attenuating a propagation path, and as a mirror, providing an additional reflecting path. We describe two algorithms that simulate radio propagation, a ray tracing algorithm that uses a discrete sample of propagation directions and a beam tracing algorithm that enumerates reflection cones defined by building walls. The two algorithms are compared experimentally and analytically. With a triangulation-based spatial data structure, both algorithms are fast enough to provide propagation simulations in a few minutes of computing time, even for large buildings.