Polygons and polyhedra

Branko Grünbaum

The theory of convex polygons, polyhedra and polytopes is a well-developed discipline. However, even the simplest results of this field have not been extended to more general objects. The main purpose of the talk is to present some of the concepts that may form a suitable starting point for an exploration of not-necessarily- convex polyhedra. The exposition centers on polyhedra, since polygons will be discussed in more detail elsewhere, and the information available on higher-dimensional polytopes is too meager.

Polyhedra are considered not as "solids", but as families of polygons (faces), satisfying certain natural conditions. If the faces of a polyhedron are simple Jordan polygons, and if the faces intersect only along common edges or vertices, the polyhedron is called acoptic. Acoptic polyhedra are closely related to maps on orientable 2-manifolds. One of the most challenging problems is the General Realizability Conjecture, which makes precise the claim that any reasonable cell-complex decomposition of any orientable 2-manifold can be realized by an acoptic polyhedron. Various recent discoveries (such as those by Szilassi, Schwörbel, Wills, Ljubić and others) of acoptic polyhedra realizing maps that cannot be realized by polyhedra with convex faces provide support for the conjecture, although there are also maps of a rather simple character that so far have not been realized.

Other challenging problems are: (i) What is the analogue for acoptic polyhedra of Steinitz's theorem for convex polyhedra? In other words, which graphs are graphs of acoptic polyhedra? (ii) The enumeration of the types of acoptic polyhedra with a given number of faces (or vertices). This depends, naturally, on the definition of "type". It is proposed to distinguish between "combinatorial equivalence" and various levels of "isomorphism", the latter taking into account the geometric features at different levels of detail. (iii) What are the appropriate extensions of Cauchy's "rigidity theorem" to acoptic polyhedra? In particular, how can the movable polyhedra be characterized?

Generalizations to polyhedra with selfintersections, possibly with selfintersecting faces, are also of interest. Again, it is convenient to distinguish several levels of generality. Part of the interest in such polyhedra stems from the most widely known examples of this kind, the Kepler-Poinsot regular polyhedra, as well as from other regular polyhedra that can be constructed using non-simple polygons; but even more relevant is the appearance of polyhedra with selfintersections as polars of many acoptic polyhedra. Some of these aspect are discussed in other venues.

Recent Progress on the Kepler Conjecture

Thomas C. Hales

The Kepler conjecture asserts that the density of a packing of spheres of equal radius in three dimensions cannot exceed that of the face-centered cubic packing.

Recently, there has been significant progress toward a proof of this conjecture, which will be described in this talk.

Two earlier approaches to the Kepler conjecture are based respectively on the Voronoi and Delaunay decompositions of space. This work combines these earlier approaches by defining a hybrid of the Voronoi and Delaunay decompositions. This hybrid appears to retain the best attributes of the earlier approaches, and manages to avoid many of their complexities.

This recent work brings methods of global nonlinear optimization to bear on the Kepler conjecture. This talk will describe these methods and how they have been applied to the Kepler conjecture.

Computational Convexity

Victor Klee

There was a discussion of the relationship of Computational Convexity to the rest of Computational Geometry. Here are some of the main points.

(1) Computational Convexity is the common meeting ground of Convex Geometry, Mathematical Programming, and Computer Science. Rather than being motivated by intrinsicaly low-dimensional problems such as those from computer graphics, its most important problems are intrinsically high- dimensional.

(2) As a consequence of (1), it seems appropriate to focus on the case of variable dimension (i.e., the dimension is part of the input) rather than on an arbitrary fixed dimension.

(3) As a consequence of high dimensions, it seems necessary at present to focus on problems concerning bodies that are *convex*. Extensions to non-convex bodies may come later, but at present the convex case provides plenty of important challenges.

(4) Since (to exaggerate slightly) no computer ever saw an irrational number, it seems appropriate to use the binary (Turing machine) model of computational complexity (supplemented when necessary by a carefully specified oracle), rather than the model of infinite-precision real arithmetic.

(5) Because of the difficulty of its problems, the present status of Computational Convexity may be described as "largely qualitative." There has been considerable success in classifying problems as to their polynomial-time solvability or NP-hardness, but very few optimal algorithms are known. The task remains to find optimal exact algorithms for problems that have been found to be solvable in polynomial time, and to find good heuristics or useful approximation algorithms for problems that are NP- hard.

(6) Significant challenges for the future are provided by several important problems that are, by some, already regarded as "well-solved" because for each *fixed* dimension there is an algorithm of low worst- case complexity. However, when the exponent or the multiplicative constant in the complexity estimate increases exponentially with the dimension, such "good" solutions, still leave room for much improvement from the viewpoint of Computational Convexity and also from the viewpoint of actual use.

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After presenting the above motivation, some results and probems from specific areas of computational convexity were presented. These appear (along with much additional material) in the following two survey articles:

P. Gritzmann & V. Klee, On the complexity of some basic problems in computational convexity: I. Containment problems, Discrete Math. 136 (1994) 129–174;

P. Gritzmann & V. Klee, On the complexity of some basic problems in computational convexity: II. Volume and mixed volumes, in Polytopes: Abstract, Convex and Computational, Kluwer, Boston, 1994, 373–466.

For more detailed studies of inner and outer j-radii, and of largest j-simplices in d-polytopes, see the following papers:

P. Gritzmann & V. Klee, Inner and outer j-radii of convex bodies in finite-dimensional normed spaces, Discrete Comput. Geom. 7 (1992) 255–280;

P. Gritzmann & V. Klee, Computational complexity of inner and outer j- radii of convex polytopes, Math. Programming A 59 (1993) 163–213;

P. Gritzmann, V. Klee & D. Larman, Largest j-simplices in n-polytopes, Discrete Comput. Geom. 13 (1995) 477–515;

M. Hudelson, V. Klee & D. Larman, Largest j-simplices in d-cubes: Some relatives of the Hadamard maximum determinant problem, Linear Algebra Appl. 241–243, 1996, 519–598.

Randomization and derandomization in computational geometry

Jiří Matoušek

In the first part, we discuss mainly randomized incremental algorithms in computational geometry. We formulate several axiomatic frameworks for such algorithms, and point out differences among them. We mention related open problems (analyzing randomized incremental constructions for highly degenerate point configurations, etc.).

In the second part, we discuss techniques for replacing randomized algorithms by deterministic ones with a similar asymptotic running time. All successfully derandomized algorithms are essentially of the divide-and-conquer type. If we allow for an n^δ extra factor ($\delta > 0$ an arbitrarily small constant) in the running time, then derandomization is usually easy using known techniques (computation of ε-nets and ε-approximations). Only partial results and techniques of a limited applicability are known for optimal derandomization, and the successful examples are fairly complex.

Combinatorics of Visibility and Illumination: 36 Open Problems

Joseph O'Rourke*

Abstract

A compendium of thirty-six open problems is presented: on point visibility, floodlight illumination, and visibility graphs.

*Department of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu. Supported by NSF grant CCR-9421670.

Extremal Problems in Combinatorial Geometry

János Pach

We survey several recent developments in combinatorial geometry. The results presented are divided into four categories. (A) Quantitative Helly-Erdős-Szekeres-Tverberg-type theorems, (B) Distribution of distances among n points, (C) Bounds on the number of incidences between points and curves or surfaces, (D) Crossing numbers of graphs.

As an example of a problem belonging to (A), we present a detailed proof of the following result. Let $P_1, \ldots, P_{d+1}$ be pairwise disjoint n-element point sets in general position in d-space. Then there exist a point O and suitable subsets $Q_i \subseteq P_i$ $(i = 1, \ldots, d+1)$ such that $|Q_i| \geq c_d|P_i|$ and every d-dimensional simplex with exactly one vertex from each Q_i contains O in its interior. Here c_d is a positive constant depending only on d. This generalizes a theorem of Vrećica and Źivaljevič. The proof is based on the Szemerédi Regularity Lemma.

Universality Theorems for Oriented Matroids and Polytopes

Jürgen Richter-Gebert

The realization space of a rank d oriented matroid $\mathcal{M}$ is the set of all point configurations $X \in \mathbf{R}^\mathbf{d}$ that have $\mathcal{M}$ as underlying oriented matroid (modulo linear equivalence). The realization space of a d-dimensional polytope P is the set of all polytopes in $\mathbf{R}^\mathbf{d}$ that have the same face lattice as P (modulo affine equivalence). The *Universality Theorem for rank 3 oriented matroids* (Mnëv, 1988) states that for every basic primary semialgebraic set V (over $\mathbf{Z}$) there is a rank 3 oriented matroid whose realization space is stably equivalent to V. The *Universality Theorem for* 4*-dimensional Polytopes* (Richter-Gebert, 1995) is a similar statement in the category of 4-dimensional polytopes.

The proof in the oriented matroid case can be performed by starting with the defining inequality system of V, performing a (non-trivial) algebraic translation into a *Shor normal form*, and then modelling elementary multiplications and additions by the classical von Staudt constructions (this is, in essence, Mnëv's construction). The proof for the case of 4-polytopes proceedes in a similar way. However the role of von Staudt constructions must be replaced by a relatively complicated polytopal construction.

We demonstrate that it is possible to derive a relatively simple proof for a Universality Theorem for 6-dimensional Polytopes by directly making use of Mnëv's construction. The proof proceeds as follows. Starting from a (basic primary) semialgebraic set V we take the corresponding point configuration $X(V)$ that is constructed by Mnëv's proof. The corresponding *zonotope* $Z = Z(X(V))$ is a 3-dimensional polytope. In the category of zonotopes the realization space of Z is stably equivalent to V. The property of Z beeing a (generalized) zonotope can be fixed by applying *connected sums* and *Lawrence extensions.* By this we end up with a 6-dimensional polytope $P = P(Z(X(V)))$ that contains Z as a 3-face. Every realization of P the face Z is projectively equivalent to a (generalized) zonotope. The realization space of P is stably equivalent to V.

Cinderella, a program for geometric drawing and automated theorem proving (demonstration)

Jürgen Richter-Gebert

Cinderella is a program for doing (projective) incidence geometry interactively on a computer. There are two main tasks for which *Cinderella* is designed:

- *Interactive drawing*

  The system provides facilities to construct geometric configurations containing points, lines and conics by performing mouse actions on a drawing surface. The construction sequence is memorized and can be changed by moving the base elements of the construction. Different ports allow to view the same configuration simultaneously under different aspects (primal, polarized, etc.).

- *Automatic proving*

  The system is equipped with an algebraic theorem prover that allows one to generate short (and readable) proofs for many projective incidence theorems. The prover is based on the method of *binomial final polynomials* that was originally developed to prove non-realizability of oriented matroids. Using invariant theoretic methods à la F. Klein the scope of theorem prover that can be extended also to euclidean, elliptic and hyperbolic geometry.

Interactions between real algebraic geometry and discrete and computational geometry

Marie-Françoise Roy
IRMAR (URA CNRS305) ,Université de Rennes,FRISCO (ESPRIT LTR)

The realization space of an order type is a basic semi-algebraic (which we abbreviate as "sa") set of R^{2n} (i.e. a set defined by a finite conjunction of sign conditions on polynomials). It looks like a very special sa set, but according to Mnev's Universality theorem, every basic sa set is (up to multiplication by an affine space) homeomorphic to the space of configurations of an order type. A consequence is that the answer to the isotopy conjecture is no, as a basic sa set may have many connected components.

Sa sets have a lot of remarkable stability and finiteness properties: Projections of sa sets are sa ; a sa set has a finite number of connected components, each of which is sa; sa sets can be finitely stratified; a sa set is sa homeomorphic to a simplicial complex; given a sa family, the set of parameters where the topological type is fixed is sa; sa families have finite VC dimension .

Because of these properties, sa sets are useful in modeling various problems. For example the Piano Movers Problem, (robot motions planning in a sa environment (roadmap construction), or the visual aspects of a sa set in space.

Explicit bounds in sa geometry are frequently based on the following result which follows from results by Petrowsky, Oleinik, Thom and Milnor: The number of connected component of a real algebraic set of degree d in k variables is $O(d)^k$. The number of connected components of sign conditions defined by a list of s polynomials in k variables of degree d on a variety of dimension k' can be proved to be $\binom{O(s)}{k'}O(d)^k$ (Basu-Pollack-Roy 1993). This is useful to get bounds on the number of isotopy types of configurations of n points in R^d and to the number of geometric permutations induced by k transversals to a family of $k+1-$separated compact, convex bodies in R^d (Goodman Pollack Wenger 1993).

In recent work, the complexity of algorithms for sa sets can be separated into an algebraic part (dependence on the degree) and a combinatorial part (dependence on the number). Also, the degrees of the polynomials output is independent of the number of polynomials. These features are useful in discrete and computational geometry, where $d = 1$, or d, k are fixed.

Finding a point in every connected components of sign conditions defined by a list of s polynomials in k variables of degree d on a variety of dimension k' can be done in time $\binom{O(s)}{k'}sd^{O(k)}$ (Basu-Pollack-Roy 1996)

Given $\Phi(Y) = (Q_\omega X^{[\omega]})\dots(Q_1X^{[1]})F(P_1,\dots,P_s)$, it is possible to compute an equivalent quantifier-free formula in $s^{(\ell+1)\Pi(k_i+1)}d^{(\ell+1)\Pi O(k_i)}$ arithmetic operations in the ring generated by the coefficients of the input polynomials. The degrees in the output are bounded by $d^{\Pi_i O(k_i)}$ which is independent of s. (Basu Pollack Roy 1994).

It is possible to compute a road map on a semi-algebraic set of dimension k' defined by s polynomials in k variables with each polynomial of degree at most d with complexity $\binom{O(s)}{k'}sd^{O(k^2)}$ (Basu-Pollack-Roy 1996).

All these results are based on the critical point method (first introduced by Grigor'ev and Vorobov) and various perturbation tricks.

Software efforts (Canny's toolkit, Rouilliers' Real Solving (POSSO FRISCO project)) are made to implement the critical point method. The challenge is to be able to decide whether or not a small (written on half a page) system of equalities and inequalities is consistent.

The central open algorithmic problems are: Compute the real dimension of a real algebraic set or of a semi algebraic set with algebraic complexity $d^{O(k)}$. Compute stratifications of sa sets in single exponential time.

Various quantities associated to sa sets can be used to give lower bounds for many problems in discrete and computational geometry concerning sorting, element distinctness (Ben-Or, Bjorner-Lovasz- Yao, Recio et al), polytope membership (Grigor'ev- Karpinsky-Vorobjov) linear algebra, sign determination (Lickteig, Lickteig-Roy). The idea is that small sa decision or computation trees cannot define sa sets with many connected components, huge sum of Betti numbers, or high geometric degrees.

Finally, the Signed Newton Diagram is a fascinating object connecting discrete geometry to the topology of real algebraic sets. Two examples have been developed: Viro's method for curves, and a conjecture generalizing Descartes's law of signs (Itenberg-Roy).

Periodic and Aperiodic Tilings

Marjorie Senechal

The rapid and extensive development of tiling theory since 1986 can be traced in large measure to the influence of "Tilings and Patterns", by Branko Grünbaum and Geoffrey Shephard. We show, in particular, how subtle aspects of the relations between local configurations in and the global structure of both periodic and aperiodic tilings are elucidated by considerations developed in that work.

Complexity of Arrangements

Micha Sharir
Tel Aviv University and New York University

We survey recent progress on the problem of bounding the combinatorial complexity of various substructures in arrangements of n algebraic surface or surface patches of constant maximum degree in $d \geq 3$ dimensions. (Informally, such an arrangement is the subdivision of space into cells of various dimensions induced by the given surfaces.) Arrangements of this kind are a central construct in computational geometry, and arise in many applications, such as motion planning in robotics, ray shooting and visibility in three dimensions, Voronoi diagrams, transversals, geometric optimization, and more. In these applications, we are usually interested not in the full arrangement (whose complexity is in general $\Theta(n^d)$), but in various portions, such as the lower or upper envelope of the surfaces, a single cell of the arrangement, a 'zone' of another surface, etc.

To show that such portions have smaller combinatorial complexity, has been a major open problem for nearly a decade, ever since analogous results have been obtained for 2-dimensional arrangements, using the theory of Davenport-Schinzel sequences (which we also briefly review). Considerable progress has been made in the past 3-4 years, where almost tight bounds (close to $O(n^{d-1})$) for the complexity of such arrangement portions have been obtained, including many related results. In the talk we survey these recent developments, demonstrate some of the analysis techniques on some simple examples, and present some combinatorial and algorithmic applications of the new bounds.

New Results on Packings in Grassmannian Space and Connections with Quantum-error-correcting Codes

N. J. A. Sloane

AT&T Research, Murray Hill, New Jersey 07974

This work began when R. H. Hardin, J. H. Conway and I set out to investigate the question: How should N n-dimensional subspaces of m-dimensional Euclidean space be arranged so that they are as far apart as possible? The results of extensive computations for modest values of N, n, m are described, as well as a reformulation of the problem that was suggested by these computations. The reformulation gives a way to describe n-dimensional subspaces of m-space as points on a sphere in dimension $(m-1)(m+2)/2$, which provides a (usually) lower-dimensional representation than the Plücker embedding, and leads to a proof that many of the new packings are optimal. The results have applications to the graphical display of multi-dimensional data via Asimov's grand tour method.

In the past few weeks some astonishing connections have been discovered between this problem and the problem of constructing quantum-error-codes, which has led to further discoveries in both fields. The latter is joint work with A. R. Calderbank, E. M. Rains and P. W. Shor.

Geometric Discrepancy Theory

Joel Spencer

Abstract

The discrepancy of a family $\mathcal{A}$ is the least K for which there *exists* a two coloring $\chi : \Omega \to \{-1, +1\}$ so that $|\chi(A)| \leq K$ for each $A \in \mathcal{A}$. Here Ω is the set of underlying vertices and $\chi(A) = \sum_{a \in A} \chi(a)$. We compare and contrast three settings: general set systems, where one gets upper bounds on the discrepancy in terms of $|\Omega|$, the $|A|$, $\deg(x)$, etc., number theory, with $\Omega = \{1, \ldots, n\}$ and $\mathcal{A}$ the family of arithmetc progressions, and our core topic geometry, with $\Omega \subset R^d$ and $\mathcal{A}$ the half spaces, balls, or whatever.

The basic Erdős Existence argument is described and it is noted that the bounds are far from best possible in the number theoretic and geometric realms. The Floating Colors method of Beck and Fiala is given and applied to axis parallel boxes in the geometric setting to give a polylogarithmic bound. A classic 1964 paper of K.F. Roth uses Fourier analysis to give a lower bound $\Omega(n^{1/4})$ in the number theoretic setting and this is contrasted with this author's version of Janos Pach's version of Bernard Chazelle's version of Alexander's 1990 lower bound of $\Omega(n^{1/4})$ for half spaces in the plane. In both cases an appropriate variance of a random set is shown to be high.

An Entropy argument inaugurated by this author and both expanded and simplified by Ravi Boppana and Jiri Matousek is given. It is shown that under certain conditions on the $|A|$ and $|\Omega|$ there exists a partial coloration $(\chi : \Omega \to \{-1, 0, +1\}$ with the $|\chi(A)|$ small and a positive proportion of the $\chi(v) \neq 0$. One can then sometimes iterate this procedure to give a good full coloring. In the initial 1985 application this author showed that any n sets on n vertices have discrepancy $O(n^{1/2})$. With Matousek this author showed in 1994 that the number theory case has discrepancy $O(n^{1/4})$ and Matousek showed that half spaces in the plane have discrepancy $O(n^{1/4})$.

Proof of Reay's conjecture on certain 1-dimensional intersections

Helge Tverberg
University of Bergen
5020 Bergen, Norway
tverberg@mi.uib.no

In 1979 John Reay conjectured (see Israel J. Math. 34 (238-244) or conjecture 9.12 in Eckhoff's chapter 2.1 of Handbook of Convex Geometry):

Let $2d(r-1)+2$ points be given in d-space. Then they can be grouped into r parts so that the intersection of the corresponding convex hulls is at least 1-dimensional.

This is proved, and the problem of finding the cases in which one cannot do with one point less, is discussed.

Geometric Transversal Theory

Dr. Rephael Wenger
Department of Computer and Information Science
The Ohio State University

A k-transversal to a family of convex sets in R^d is an affine subspace of dimension k (such as a point, line, plane or hyperplane) which intersects every member of the family. This talk covers some of the major results in the past ten years on k-transversals and poses open questions related to these major results. In particular, I discuss necessary and sufficient conditions for the existence of transversals, Tverberg's Helly-type theorem for line tranversals of translates in the plane, piercing or Gallai numbers, the combinatorial complexity and the topological structure of the space of transversals, the order in which transversals intersect a family of sets, and a theory of convexity on the affine Grassmannian developed by Goodman and Pollack.

Sphere Packing and Crystal Growth

Let B^d denote the unit ball in euclidean d–space E^d, $d \geq 2$. For a convex body $K \subset E^d$ let $V(K)$ denote its volume. For a finite set $C_n = \{c_1, \dots, c_n\} \subset E^d$ with $||c_i - c_j|| \geq 2$, $i \neq j$ we call $C_n + B^d$ a finite sphere packing or briefly a packing. If $L \subset E^d$ is a packing lattice for B^d and $C_n \subset L$, then $C_n + B^d$ is a lattice packing. For $\rho > 0$ the parametric density of $C_n + B^d$ is defined by

$$\delta_\rho(B^d, C_n) = nV(B^d)/V(\operatorname{conv} C_n + \rho B^d).$$

Further

$$\delta_\rho(B^d, n) = \max\{\delta_\rho(B^d, C_n) | C_n + B^d \text{ packing}\}.$$

The optimal C_n with $\delta_\rho(B^d, C_n) = \delta_\rho(B^d, n)$ are denoted by $C_{n,\rho}$.
If $\operatorname{conv} C_n$ is a segment of length $2(n-1)$, we write $C_n = S_n$ and call $S_n + B^d$ a sausage.
For each $d \geq 2$ and each $n \geq 3$ there is a critical radius $\rho_{d,n}$ with $C_{n,\rho} = S_n$ for $0 < \rho \leq \rho_{d,n}$ and $C_{n,\rho} \neq S_n$ for $\rho > \rho_{d,n}$.
The parametric density δ, the critical radii $\rho_{d,n}$, the optimal shapes $C_{n,\rho}$ and their analogues δ^L_ρ, $\rho^L_{d,n}$, $C^L_{n,\rho}$ for restriction to lattice packings are the main objects of finite sphere packings. One can show their asymptotic close relation to classical sphere packing, lattice and nonlattice. One can further show how the classical theory fits together with the typical phenomena of parametric density as sausages and sausage catastrophes.

Here we show that for large n and suitable ρ the normalized $C^L_{n,\rho}$, i.e. $n^{-1/d} \operatorname{conv} C^L_{n,\rho}$ tend to Wulff–shapes, i.e. to shapes of real crystals. If one replaces $V(\operatorname{conv} C_n + \rho B^d)$ by $V(\operatorname{conv} C_n + \rho C)$ with a suitable convex body C, one can control unisotropies, which occur in crystals by bonds. This gives a much richer variety of shapes. Even extreme shapes of crystals (e.g. whiskers) can be realized via parametric density.

J.M. Wills (Siegen)

Günter M. Ziegler
(TU Berlin)

Recent Progress on Polytopes

We outline five very active and important areas of research concerned with the (combinatorial) theory of (convex) polytopes, report about recent progress, and present five "challenge" problems that we hope to see solved soon.

Universality Theorems for polytopes of constant dimension: see Richter-Gebert's recent work! *Challenge:* Provide a Universality Theorem for simplicial polytopes of constant dimension.

Triangulations and subdivisions of polytopes. *Challenge:* Decide whether all triangulations on a fixed point set in general position can be connected by bistellar flips.

0/1-polytopes and their combinatorial structure. *Challenge:* Are the numbers of facets of 0/1-polytopes bounded by an exponential function in the dimension?

Neighborly polytopes. Explicit constructions and extremal properties.

Paths. The existence and construction of short monotone paths to the "top vertex" of a polytope is a crucial problem in the analysis of the simplex algorithm for linear programming. We report about joint work with Nina Amenta on "deformed products" of polytopes, a new construction that all the known examples of linear programs with exponential lower bounds. *Challenge:* Decide whether there is a polynomial upper bound for the expected running time of the RANDOM-EDGE simplex algorithm. *Challenge:* The "Monotone Upper Bound Problem": What is the maximal number of vertices of a monotone path on a d-dimensional polytope with n facets?