Discrete and Computational Geometry: Ten Years Later, Mount Holyoke College.
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(Collection of open problems in order of presentation, edited by E. Welzl)

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Does there exist a "strong Einstein"? More precisely: Is there a tile T in $\mathbb{E}^d$ or $\mathbb{H}^d$ with the following properties:

- *int* (T) connected,
- $T = cl\ (int\ (T))$,
- T is compact,
- T does tile the space,
- if P is a T-tiling, then the symmetry group $\mathrm{Sym}(P)$ is finite;

or – somewhat weaker –

- $\mathrm{Sym}(P)$ is finite for every facet-to-facet T-tiling P (∂T dissected into "facets" forming a cell-complex)?

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Let P be a set of n points on a horizontal line l in $\mathbb{R}^3$ labelled $p_1, p_2, \ldots p_n$ although not necessarily in order. We wish to connect the points in the order $p_1, p_2, p_3, \ldots p_n$ by a simple curve (not self-interseetings) which has minimum number of intersections with the line l. In the worst case, $\Theta(n^2)$ intersections are necessary and sufficient.

Give a polynomial time algorithm to find the optimal solution or prove the problem is NP-complete or give an approximation algorithm.

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Determine the smallest number r that has the following property: Whenever a 3-dimensional convex body can be passed through a circle of radius 1 (with "turning" permitted), it can also be passed through a long circular cyclinder of radius r. (It is known that $r > 1$.)

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Let S be a set of n lines in the plane and $T = \bigcup S$ be their union. Let p and q be points in T. Given S, can one determine in $o(n^2)$ time, the shortest (Euclidean) path in T from p to q?

Using a real RAM model of computation that can compute distances, $O(n^2)$ time can be achieved by constructing the arrangement and using standard techniques. The problem is to find and exploit some metric structure in an arrangement of lines to obtain a subquadratic algorithm. Simple arguments lead to an $O(n \log n)$ algorithm for a path that is within factor of 2 of the shortest. (See [Bose et al., Canadian Conf. Comp. Geom. 1996])

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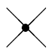
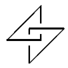
The fact that the graph $K_{3,3}$ is not planar can be abbreviated as follows

$$(K_{3,3} \rightarrow \mathbb{R}^2) \implies (2 \mapsto 0\text{-}dim) \qquad or$$

$$(K_{3,3} \rightarrow \mathbb{R}^2) \implies$$ 

where $2 \mapsto 0\text{-}dim$ means that the images of some two independent edges of $K_{3,3}$ intersect, i.e. have a common 0-dimensional transversal.

It is also known that

$$(K_6 \hookrightarrow \mathbb{R}^3) \implies$$ 

which means that for any embedding $K_6 \hookrightarrow \mathbb{R}^3$, there exist two linked triangles in the image, [1].

Problem: Find the characterization in terms of forbidden graphs (minors) of all graphs G with the property

$$(G \hookrightarrow \mathbb{R}^3) \implies (4 \mapsto 1\text{-}dim)$$

where $4 \mapsto 1\text{-}dim$ means that some four independent edges of G have a common line transversal.

[1] P. Seymour, Progress on the four-color theorem, *Proc. Int. Cong. Math. Zürich, Switzerland, Birkhäuser,* 1995.

[2] R. Živaljević, The Tverberg-Vrećica problem and the Combinatorial Geometry on vector bundles, *preprint.*

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How many non-simplex facets can an n-vertex 4-polytope have?

Examples with $\Omega(n)$ non-simplex facets are easy to construct, and an upper bound of $O(n^2)$ follows immediately from McMullen's upper bound theorem. As far as I know, these are the tightest known bounds, even for polyhedral 3-spheres.

The best construction I know of is the connected sum of $n/5 - 1$ copies of a bipyramid over a cube, which has n vertices and $11n/5 - 10$ facets, each a square pyramid. Is $3n$ achievable?

There is an asymptotically tight bound of $\Theta(n^2)$ in five dimensions. The lower bound is achieved by the convex hull of a set of integer points on the "weird moment curve" (t, t^2, t^3, t^4, t^6). For example, when n is even, the set $\{-n, 2-n, \ldots, -4, -2, 1, 2, \ldots, n/2\}$ induces a polytope with n vertices and $n^2/16 + \Theta(n)$ bipyramidal facets.

More generally, in d dimensions, we have an upper bound of $O(n^{\lfloor d/2 \rfloor})$, and a generalization of the weird moment curve construction gives us a lower bound of $\Omega(n^{\lceil n/2 \rceil - 1})$. Which bound is correct when d is even?

I am particularly interested in the case of *quasisimplicial* polytopes, each of whose $(d-2)$-faces is a simplex. The weird moment curve examples are quasisimplicial. See [J. Erickson. New lower bounds for convex hull problems in odd dimensions. In *Proc. 12th Ann. ACM Symp. Comptuational Geometry*, pp. 1–9, 1996].

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Is there an infinite family of combinatorial types of 4-polytopes P with the property that the sum of the number of vertices and facets of P is asymptotically smaller than the sum of the faces of other dimensions, i.e.

$$f_0(P) + f_3(P) = o(f_1(P) + f_2(P))\,?$$

Such classes of "fat-lattice" polytopes are known to exist for dimension $d \geq 6$. For dimension $d = 3$ such classes cannot exist because of Euler's relation.

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Are there two triangulations of the 7-simplex without a common stellar subdivision?

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Given a family $\mathcal{L}$ of directed lines through the origin in $\mathbb{R}^d$, no two orthogonal, its acuteness graph $\mathcal{A}(\mathcal{L})$ (as defined in joint work with Boris Aronov and Richard Pollack) is the graph whose vertex set is $\mathcal{L}$ in which two vertices are joined by an edge if and only if the corresponding directed lines make an acute angle. It is not difficult to see that every graph G can be realized as the acuteness graph of some family of directed lines, and the lowest dimension d such that the lines can be chosen to lie in $\mathbb{R}^d$ is called the *acuteness number* $\alpha(G)$ of G.

If C_n is the n-cycle (the graph with n vertices joined in cyclic order), it is not hard to see that $\alpha(C_3) = 1$, $\alpha(C_5) = \alpha(C_6) = \alpha(C_7) = 2$, $\alpha(C_4) = \alpha(C_8) = 3$, and $\alpha(C_{10}) = 4$, and that in general $\alpha(C_{2d+2}) = d$. But the value of $\alpha(C_n)$ is not clear for odd $n \geq 9$ (it is either $(n-1)/2$ or $(n-3)/2$, so that $\alpha(C_n)$ is monotone for $n \geq 5$), and the problem we propose is to determine (at least) $\alpha(C_9)$.

Comment: After the problem was presented, Peter Shor and Egon Schulte independently worked out the value of $\alpha(C_n)$ for n even.

Marshall Bern (Problem from Steve Vavasis) `bern@parc.xerox.com`

Let $P = s_1 s_2 \ldots s_n$ be a simple polygon in the Euclidean plane. Assume you are given the following information about P: (1) for each i, the measure of the angle at s_i, (2) the "names" of all the diagonals in a triangulation of P, that is, $d_1 = s_{i_1} s_{j_1}$, $d_2 = s_{i_2} s_{j_2}$, $\ldots$, $d_{n-3} = s_{i_{n-3}} s_{j_{n-3}}$, and (3) for each d_i, the cross-ratio of the quadrilateral formed by the two triangles bounded by d_i. The cross-ratio of a quadrilateral is the product of the lengths of one pair of opposite sides divided by the product of the lengths of the other pair. (For specificity, we may assume that the numerator is always the product of the two sides clockwise from d_i.)

PROBLEM: Does this information uniquely determine P, up to similarity?

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Given finite $S \subset \mathbb{R}^2$, a point $p \in \mathbb{R}^2$ is an S-maximum if $p \in S$ and there is no $q \in S$, $q \neq p$, such that $p_x \leq q_x$ and $p_y \leq q_y$. Given $q \in \mathbb{R}^2$, let $S_q = \{\, p \in S \mid p_x \leq q_x \text{ and } p_y \leq q_y \,\}$. Let $B(S, q)$ denote the number of S_q-maxima. Finally, let $B(S) = \max_q B(S, q)$.

PROVE OR DISPROVE: For any $S \subset \mathbb{R}^2$, $|S| = n$, there exists $T \subset \mathbb{R}^2$, $|T| = O(n)$ such that $B(S \cup T) = O(\log n)$.

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PROVE OR DISPROVE the following generalization of the Sylvester-Gallai Theorem: Given n points in the plane not all of which are colinear, and given any balanced non-Radon partition (P, Q) of the point set, there exists an ordinary line containing exactly one point from each of P and Q?

Note. A partition (P, Q) is called non-Radon if there is a line strongly separating P and Q, and is balanced if their sizes differ at most by one. It is not difficult to see that neither "balanced" nor "non-Radon" can be eliminated for generalization.

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PROBLEM: Are there positive constants c_1, c_2, such that every set P of n points in the plane allows a matching of size $c_1\sqrt{n}$ with no line crossing more than c_2 edges of the matching? (A line *crosses* an edge $\{p, q\}$, if the points p and q lie on opposite sides of the line.)

WHAT IS KNOWN? It is always possible to choose a matching of size $\sqrt{n}$ with no line crossing more than $O(\log n / \log\log n)$ edges. And for any fixed positive $\varepsilon \leq 1/2$, there is a matching of size $n^{(1/2)-\varepsilon}$ with no line crossing more than $O(1)$ edges. (See [E. Welzl, On spanning trees with low crossing numbers, *Lecture Notes in Computer Science* **594** (1991) 233-249].) An old result by Erdős and Szekeres on subsets in convex position [*Compositio Mathematica* **2** (1935) 463-470], gives a matching of size $\Theta(\log n)$ with every line crossing at most two edges. Is $\Theta(\log n)$ optimal for "2"?