Chapter 3

Convex Hull Problems

The construction of convex hulls is perhaps the oldest and best-studied problems in computational geometry [6, 10, 11, 12, 29, 28, 30, 36, 49, 50, 91, 101, 110, 123, 130, 132, 134, 136, 142]. Over twenty years ago, Graham described an algorithm that constructs the convex hull of n points in the plane in $O(n \log n)$ time [91]. The same running time was first achieved in three dimensions by Preparata and Hong [123]. Yao [154] proved a lower bound of $\Omega(n \log n)$ on the complexity of identifying the convex hull vertices, in the quadratic decision tree model. This lower bound was later generalized to the algebraic decision tree and algebraic computation tree models by Ben-Or [16]. It follows that both Graham's scan and Preparata and Hong's algorithm are optimal in the worst case. If the output size f is also taken into account, the lower bound drops to $\Omega(n \log f)$ [101], and a number of algorithms match this bound both in the plane [101, 28, 29] and in three dimensions [50, 40].

In higher dimensions, the problem is not quite so completely solved. Seidel's "beneath-beyond" algorithm [132] constructs d-dimensional convex hulls in time $O(n^{\lceil d/2 \rceil})$. After a ten-year wait, Chazelle [36] improved the running time to $O(n^{\lfloor d/2 \rfloor})$ by derandomizing a randomized incremental algorithm of Clarkson and Shor [50]; see also [136]. Since an n-vertex polytope in $\mathbb{R}^d$ can have $\Omega(n^{\lfloor d/2 \rfloor})$ facets [87], Seidel's algorithm is optimal in even dimensions, and Chazelle's algorithm is optimal in all dimensions, in the worst case.

Several faster algorithms are known when the output size is also considered. In 1970, Chand and Kapur [30] described an algorithm that constructs convex hulls in time $O(nf)$, where f is the number of facets in the output. An algorithm of Chan, Snoeyink, and Yap [28] constructs four-dimensional hulls in time $O((n + f) \log^2 f)$, and a recent improvement by Amato and Ramos [6] constructs five-dimensional hulls in time $O((n+f)\log^3 f)$. The fastest algorithm in higher dimensions, due to Chan [29], runs in time $O(n \log f + (nf)^{1-1/(\lfloor d/2 \rfloor+1)} \operatorname{polylog} n)$; this algorithm is optimal when f is sufficiently small. For related results, see [10, 30, 49, 50, 101, 134]. There are still large gaps between these upper bounds and the lower bound $\Omega(n \log f + f)$. Avis, Bremner, and Seidel [11, 12] describe families of polytopes on which current convex hull algorithms perform quite badly, sometimes requiring exponential time (in d) even when the output size is only polynomial.

In this chapter, we consider convex hull problems for which the output size is a single integer, or even a single bit, although the convex hull itself may be large. We show that in the worst case, $\Omega(n^{\lceil d/2 \rceil - 1} + n \log n)$ sidedness queries are required to decide whether the convex hull of n points in $\mathbb{R}^d$ is simplicial, or to determine the number of convex hull facets. This matches known upper bounds when d is odd [36]. The only lower bound previously known for either of these problems is $\Omega(n \log n)$, following from the techniques of Yao [154] and Ben-Or [16]. When the dimension is allowed to vary with the input size, deciding if a convex hull is simplicial is coNP-complete [31], and counting the number of facets is #P-hard in general, and NP-hard for simplicial polytopes [61]. Our results apply when the dimension d is fixed.

Our lower bounds follow from a generalization of the previous chapter's adversary argument. We start by constructing a set whose convex hull contains a large number of independent degenerate facets. To obtain the adversary configuration, we perturb this set to eliminate the degeneracies, but in a way that the degeneracies are still "almost there". An adversary can reintroduce any one of the degenerate facets, by moving its vertices back to their original position.

3.1 Preliminaries

For a more detailed introduction to the theory of convex polytopes, we refer the reader to Ziegler [161] or Grünbaum [95].

The *convex hull* of a set of points is the smallest convex set that contains it. A *(convex) polytope* is the convex hull of a finite set of points. A hyperplane h *supports* a polytope if the polytope intersects h and lies in a closed halfspace of h. The intersection of a polytope and a supporting hyperplane is called a *face* of the polytope. The *dimension* of a face is the dimension of the smallest affine space that contains it; a face of dimension k is called a k-*face*. The faces of a polytope are also polytopes. Given a d-dimensional polytope, its $(d-1)$-faces are called *facets*, its $(d-2)$-faces are called *ridges*, its 1-faces are *edges*, and its 0-faces are *vertices*.

A polytope is *simplicial* if all its facets, and thus all its faces, are simplices. The convex hull of any affinely nondegenerate set of points is simplicial, but the converse is not true in general, as witnessed by the regular octahedron in $\mathbb{R}^3$. A polytope is *quasi-simplicial* if all of its ridges are simplices, or equivalently, if its facets are simplicial polytopes. A *degenerate facet* of a quasi-simplicial polytope is any facet that is not a simplex. Note that the vertices of a degenerate facet are also the vertices of a degenerate simplex.

3.2 The Lower Bound

Our adversary construction will consist of a set of points on the weird moment curve $\omega_d(t) = (t, t^2, \dots, t^{d-1}, t^{d+1})$ introduced in Section 2.2. Since any collection of points on the standard moment curve is in convex position, so is any collection of points on the weird moment curve in dimensions 3 and higher. Moreover, the convex hull of any set of points on the weird moment curve is quasi-simplicial, since no d points lie on a common $(d-2)$-flat. However, degenerate facets are possible. The following lemma characterizes degenerate convex hull facets on the weird moment curve. The result is quite similar to Gale's evenness condition [87], which describes which vertices of a cyclic polytope form its facets.

Lemma 3.1. *Let X be a set of real numbers, and let $x_0, x_1, \dots, x_d$ be elements of X whose sum is zero. The points $\omega_d(x_0), \omega_d(x_1), \dots, \omega_d(x_d)$ are the vertices of a degenerate facet of $\operatorname{conv}(\omega_d(X))$ if and only if for any two elements $y, z \in X \setminus \{x_0, x_1, \dots, x_d\}$, the number of elements of $\{x_0, x_1, \dots, x_d\}$ between y and z is even.*

Proof: Let h be the hyperplane passing through the points $\omega_d(x_0), \omega_d(x_1), \dots, \omega_d(x_d)$. Such a hyperplane exists by Lemma 2.3. Expanding the appropriate sidedness determinant, we find that an arbitrary point $\omega_d(x)$ lies above, on, or below h according to the sign of the polynomial

$$f(x) = \left(x + \sum_{i=1}^{d} x_i\right) \prod_{i=1}^{d} (x - x_i) = \prod_{i=0}^{d} (x - x_i).$$

The hyperplane h supports $\mathrm{conv}(\omega_d(X))$ if and only if $f(x)$ has the same sign for all $x \in X \setminus \{x_0, x_1, \ldots, x_d\}$.

The polynomial $f(x)$ has degree $d+1$, and vanishes at each x_i. Thus, the sign of $f(x)$ changes at each x_i. In more geometric terms, the weird moment curve crosses the hyperplane h at each of the points $\omega_d(x_i)$. It follows that $f(y)$ and $f(z)$ both have the same sign if and only if an even number of x_i's lie between y and z. □

The main result of this chapter is based on the following combinatorial construction.

Lemma 3.2. *For all n and d, there is a quasi-simplicial polytope in $\mathbb{R}^d$ with $O(n)$ vertices and $\Omega(n^{\lceil d/2 \rceil - 1})$ degenerate facets.*

Proof: First consider the case when d is odd, and let $r = (d-1)/2$. Without loss of generality, we assume that n is a multiple of r. Let X denote the following set of $n + 2n/r = O(n)$ integers.

$$X = \{-rn, -rn + r, \ldots, -r;\ r, r+1, 2r, 2r+1, \ldots, n, n+1\}$$

We can specify a degenerate facet of $\omega_d(X)$ as follows. Arbitrarily choose r elements $a_1, a_2, \ldots, a_r \in X$, all positive multiples of r. Let $a_0 = -\sum_{i=1}^{r} a_i$, let $b_0 = a_0 - r$, and for all $i > 0$, let $b_i = a_i + 1$. Each a_i and b_i is a unique element of X, and no element of X lies between a_i and b_i for any i. The points $\omega_d(a_i)$ and $\omega_d(b_i)$ all lie on a single hyperplane by Lemma 2.3, since

$$\sum_{i=0}^{r} (a_i + b_i) = 2 \sum_{i=0}^{r} a_i = 0.$$

Moreover, since any pair of elements of $X \setminus \{a_i, b_i\}$ has an even number of elements of $\{a_i, b_i\}$ between them, Lemma 3.1 implies that these points are the vertices of a single facet of $\mathrm{conv}(\omega_d(X))$. There are at least $\binom{n/r}{r} = \Omega(n^r)$ ways of choosing such a degenerate facet.

When d is even, let $r = d/2 - 1$, and assume without loss of generality that n is a multiple of r. Let X be the following set of $n + 2n/r + 1 = O(n)$ integers.

$$X = \{-n - rn, -n - rn + r, \ldots, -n - r;\ r, r+1, 2r, 2r+1, \ldots, n, n+1;\ 2n\}.$$

Using similar arguments as above, we easily observe that the polytope $\mathrm{conv}(\omega_d(X))$ has $\Omega(n^r)$ degenerate facets, each of which has $\omega_d(2n)$ as a vertex. □

This result is the best possible when d is odd, since an odd-dimensional n-vertex polytope has at most $O(n^{(d-1)/2})$ facets [161]. In the case where d is even, the best known upper bound is $O(n^{d/2})$, which is a factor of n bigger than the result we prove here. The convex hull of any set of n points on ω_d has at most $O(n^{\lceil d/2\rceil-1})$ degenerate facets, so the lower bound is tight for points on the weird moment curve. We conjecture that our lower bound is tight in general, up to constant factors.

Theorem 3.3. *Any decision tree that decides whether the convex hull of a set of n points in $\mathbb{R}^d$ is simplicial, using only sidedness queries, must have depth $\Omega(n^{\lceil d/2\rceil-1} + n\log n)$.*

Proof: Let X be the set of numbers described in the proof of Lemma 3.2, and let $X' = X + 1/(2d+2)$. Initially, the adversary presents the set of points $\omega_d(X')$. Since $\sum_{i=0}^{d} x_i'$ is always a half-integer, this point set is affinely nondegenerate, so its convex hull is simplicial.

It suffices to consider the case where d is odd. Let $r = (d-1)/2$. Choose $a_0', b_0', a_1', b_1', \ldots, a_r', b_r' \in X'$ so that $\sum_{i=0}^{r}(a_i' + b_i') = 1/2$ and no other elements of X' lie between a_i' and b_i' for any i. The corresponding points $\omega_d(a_i'), \omega_d(b_i')$ form a collapsible simplex. To collapse it, the adversary simply moves the points back to their original positions in $\omega_d(X)$. Lemmas 2.3 and 3.2 imply that the collapsed simplex forms a degenerate facet of the new convex hull. Since the sum of any other $(d+1)$-tuple changes by at most $1/2 - 1/(2d+2)$, no other simplex changes orientation. In other words, the only way for an algorithm to distinguish between the original configuration and the collapsed configuration is to perform a sidedness query on the collapsible simplex.

Thus, if an algorithm does not perform a separate sidedness query on every collapsible simplex, then the adversary can introduce a degenerate facet that the algorithm cannot detect. There are $\Omega(n^{\lceil d/2\rceil-1})$ collapsible simplices, one for each degenerate facet in $\mathrm{conv}(\omega_d(X))$.

Finally, the $n\log n$ term follows from the algebraic decision tree lower bound of Ben-Or [16]. □

A three-dimensional version of our construction is illustrated in Figure 3.1. (See also the proof of Theorem 4.3!)

Our lower bound matches known upper bounds when d is odd [36]. We emphasize that if the points are known *in advance* to lie on the weird moment curve, this problem can be solved in $O(n^{\lceil d/4\rceil})$ time if $\lceil d/2\rceil$ is odd, and in $O(n^{\lceil d/4\rceil}\log n)$ time if $\lceil d/2\rceil$ is even,

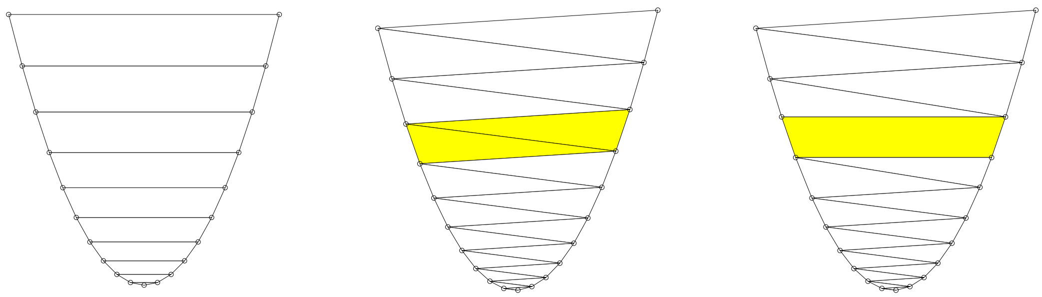
(a) (b) (c)

Figure 3.1. The convex hull adversary construction in three dimensions. Bottom views of (a) a quasi-simplicial polytope with $\Omega(n)$ degenerate facets, (b) the simplicial adversary polytope with one collapsible simplex highlighted, and (c) the corresponding collapsed polytope.

by an algorithm that uses more complicated queries, similar to the algorithm described in [73].

The convex hull of the adversary configuration $\omega_d(X')$ has $\lceil d/2 \rceil - 1$ more facets than the convex hull of any collapsed configuration. Thus, we immediately have the following lower bound.

Theorem 3.4. *Any decision tree that computes the number of convex hull facets of a set of n points in $\mathbb{R}^d$, using only sidedness queries, must have depth $\Omega(n^{\lceil d/2 \rceil - 1} + n \log n)$.*

A simple modification of our argument implies the following "output-sensitive" version of our lower bound.

Theorem 3.5. *Any decision tree that decides whether the convex hull of a set of n points in $\mathbb{R}^d$ is simplicial or computes the number of convex hull facets, using only sidedness queries, must have depth $\Omega(f)$ when d is odd, and $\Omega(f^{1-2/d})$ when d is even, where f is the number of faces of the convex hull.*

Proof: We construct a modified degenerate polytope as follows. We start by constructing a degenerate polytope with f faces, exactly as described in the proof of Lemma 3.2. When d is odd, this polytope is the convex hull of $\Theta(f^{2/(d-1)})$ points on the wired moment curve, and has $\Omega(f)$ degenerate facets. When d is even, the polytope is the convex hull of $\Theta(f^{2/d})$ points and has $\Omega(f^{1-2/d})$ degenerate facets.

By introducing a new vertex extremely close to the relative interior of any facet of a simplicial polytope, we can split that facet into d smaller facets. Each such split increases the number of polytope faces by $2^d - 2$. To bring the number of vertices of our adversary polytope up to n, we choose some facet and repeatedly split it in this fashion, being careful not to introduce any new degenerate simplices. The augmented polytope has at most $f + (2^d - 2)n = O(f)$ faces.

To get a modified *adversary* polytope, we slide the original vertices of the degenerate polytope along the weird moment curve, just enough to remove the degeneracies, leaving the new vertices in place. Each of the degenerate facets becomes a collapsible simplex. As long as we don't slide the vertices too far, collapsing a simplex will not change the orientation of any simplex involving a new vertex. (In effect, we are treating sidedness queries involving new vertices as "allowable" queries; see below.) The lower bound now follows from the usual adversary argument. □

Finally, we note that our convex hull lower bounds still hold if we augment our model of computation with extra queries as in Section 2.3. Let X be the set of numbers described in the proof of Lemma 3.2. We call an algebraic query *allowable* if for some projective transformation ϕ, the configuration $\phi(\omega_d(X))$ is nondegenerate with respect to that query.

Theorem 3.6. *Any decision tree that decides whether the convex hull of a set of n points in $\mathbb{R}^d$ is simplicial, using only sidedness queries and a finite number of allowable queries, requires $\Omega(n^{\lceil d/2 \rceil - 1} + n \log n)$ sidedness queries in the worst case.*

The proof of this theorem follows the proof of Theorem 2.6 almost exactly. The only difference is that we must consider only projective transformations that preserve the convex hull structure of $\omega_d(X)$. Alternately, we can use Stolfi's two-sided projective model, in which projective maps preserve (or reverse) the orientation of every simplex in $\mathbb{R}^d$, and thus always preserve the combinatorial structure of convex hulls; see [140, Chapter 14].

3.3 Real Convex Hull Algorithms

A large number of convex hull algorithms rely (or can be made to rely) exclusively on sidedness queries. These include the "gift-wrapping" algorithms of Chand and Kapur [30] and Swart [142], the "beneath-beyond" method of Seidel [132], Clarkson and Shor's [50] and Seidel's [136] randomized incremental algorithms, Chazelle's worst-case optimal algorithm [36], and the recursive partial-order algorithm of Clarkson [49].

Seidel's "shelling" algorithm [134] and the space-efficient gift-wrapping algorithms of Avis and Fukuda[1] [10] and Rote [130] require only sidedness queries and "second-order" coordinate comparisons between vertices of the dual hyperplane arrangment. Matoušek [110] and Chan [29] improve the running times of these algorithms (in an output-sensitive sense), by finding the extreme points more quickly. Clarkson [49] describes a similar improvement to a randomized incremental algorithm. Since every point in our adversary configuration is extreme, our lower bound still holds even if the extremity of a point can be decided for free. We are not suggesting that the computational primitives used by these algorithms cannot be used to break our lower bounds; only that the ways in which these primitives are currently applied are inherently limited.

Chan [29] describes an improvement to the gift-wrapping algorithm, using ray shooting data structures of Agarwal and Matoušek [4] and Matoušek and Schwarzkopf [108] to speed up the pivoting step. In each pivoting step, the gift-wrapping algorithm finds a new facet containing a given ridge of the convex hull. In the dual, this is equivalent to shooting a ray from a vertex of the dual polytope along one of its outgoing edges. The dual vertex that the ray hits corresponds in the primal to the new facet. A single pivoting step tells us the orientation of $n-d$ simplices, all of which share the d vertices of the new facet. However, at most one of these simplices can be collapsible, since two collapsible simplices share at most $d/2$ vertices. Thus, even if we allow a pivoting step to be performed in constant time, our lower bound still holds.

There are a few convex hull algorithms which seem to fall outside our framework, most notably the divide-prune-and-conquer algorithm of Chan, Snoeyink, and Yap [28] and its improvement by Amato and Ramos [6]. The two-dimensional version of their algorithm uses sidedness queries, along with first-, second-, and even *third*-order comparisons; higher-dimensional versions use even more complex primitives.

3.4 Open Problems

Several open problems remain to be answered. While our lower bounds match existing upper bounds in odd dimensions, there is still a gap when the dimension is even. A first step in improving our lower bounds might be to improve the combinatorial bounds in Lemma 3.2. Is there a quasi-simplicial 4-polytope with n vertices and $\Omega(n^2)$ degenerate

[1] at least if Bland's pivoting rule is used

facets? Simple variations on the weird moment curve will not suffice, since an "evenness condition" like Lemma 3.1 always forces the number of degenerate facets to be linear. Arguments based on merging facets of cyclic or product polytopes also fail, as do variations of Amenta and Ziegler's deformed products [7, 8]. I conjecture that the answer is **no**, even for polyhedral 3-spheres.

A common application of convex hull algorithms is the construction of Delaunay triangulations and Voronoi diagrams. Are $\Omega(n^{\lceil d/2 \rceil})$ in-sphere queries required to decide if the Delaunay triangulation is simplicial (*i.e.*, really a triangulation)? Again, a first step might be to construct a three- or four-dimensional Delaunay triangulation with $\Omega(n^2)$ independent degenerate features. I conjecture, however, that no such triangulations exist.

Another similar problem is deciding, given a set of points, which ones are vertices of the set's convex hull. This problem can be decided in $O(n^2)$ time (using only sidedness queries!) by invoking a linear programming algorithm once for each point [48, 109, 113, 136]. This upper bound can be improved to $O(n^{2\lfloor d/2 \rfloor/(\lfloor d/2 \rfloor - 1)} \operatorname{polylog} n)$ using an algorithm due to Chan [29]. Except for the polylogarithmic term, this algorithm is almost certainly optimal. It seems unlikely that a collapsible simplex argument could be used to imply a reasonable lower bound for this problem. Perhaps the techniques we describe in Part II are more applicable.

The suggestions described at the end of the previous chapter apply to convex hull problems as well. Richter-Gebert's universality theorem for 4-polytopes [127, 129] implies that it is NP-hard to decide if a given combinatorial 3-sphere is realizable as a 4-polytope. (In contrast, Steinitz' Theorem [161, Chapter 4] implies that every 2-sphere is realizable as a 3-polytope.) Perhaps deciding if a 4-polytope is simplicial requires $\Omega(n^2)$ time because (1) deciding if a combinatorial 3-sphere is simplicial requires quadratic time, and (2) it is not possible for an efficient algorithm to know that its input is a 4-polytope and not an arbitrary combinatorial 3-sphere. Again, this suggestion needs to be formalized before there is any hope of proving or disproving it.

Everything should be made as simple as possible,
but no simpler.

— Albert Einstein