ARBITRARILY LARGE NEIGHBORLY FAMILIES OF CONGRUENT SYMMETRIC CONVEX 3-POLYTOPES

JEFF ERICKSON[1] Department of Computer Science,
University of Illinois, Urbana, IL 61820

SCOTT KIM P.O. Box 2499, El Granada, CA 94018

Abstract. We construct, for any positive integer n, a family of n congruent convex polyhedra in $\mathbb{R}^3$, such that every pair intersects in a common facet. Our polyhedra are Voronoi regions of evenly distributed points on the helix $(t, \cos t, \sin t)$. The largest previously published example of such a family contains only eight polytopes. With a simple modification, we can ensure that each polyhedron in the family has a point, a line, and a plane of symmetry. We also generalize our construction to higher dimensions and introduce a new family of cyclic polytopes.

1. Introduction and History

A family of d-dimensional convex polytopes is *neighborly* if every pair of polytopes has a $(d-1)$-dimensional intersection. It has been known for centuries that a neighborly family of convex polygons (or any other connected sets) in the plane has at most four members. In 1905, Tietze [27, 28] proved that there are arbitrarily large neighborly families of 3-dimensional polytopes, answering an open question of Guthrie [17] and Stäckel [26]. Tietze's result was independently rediscovered by Besicovitch [4], using a different construction, and generalized to higher dimensions by Rado [11] and Eggleston [22].

Neighborly families of convex bodies are closely related to neighborly convex polytopes. A polytope is (2-)neighborly if every pair of vertices lies on

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a convex hull edge; the Schlegel diagram of the polar dual of any neighborly 4-polytope consists of a neighborly family of 3-polytopes. Neighborly polytopes were discovered by Carathéodory [7], who showed that the convex hull of any finite set of points on either the moment curve $(t, t^2, t^3, \ldots, t^d) \in \mathbb{R}^d$ or the trigonometric moment curve $(\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt) \in \mathbb{R}^{2k}$ is a neighborly polytope. Carathéodory's proof was later simplified by Gale [13], who called these polytope families the *cyclic polytopes* and the *Petrie polytopes*, respectively, and showed that the two families are combinatorially equivalent. Cyclic polytopes were independently rediscovered by Motzkin [21, 16] and Šaškin [23], among others. For further discussion of neighborly and cyclic polytopes, see Grünbaum [15] and Ziegler [32].

Dewdney and Vranch [10] showed that the Voronoi diagram of the integer points $\{(t, t^2, t^3) \mid t = 1, 2, \ldots, n\}$ form a neighborly family of unbounded convex polyhedra. Klee [18] derived a similar result for any set of evenly distributed points on the trigonometric moment curve in even dimensions four and higher. Seidel [25] observed that for any $d \geq 3$, Descartes' rule of signs[2] implies that any finite set of points on the positive branch of the d-dimensional (polynomial) moment curve has a neighborly Voronoi diagram. More generally, the vertices of any neighborly polytope have a neighborly Voronoi diagram, since the endpoints of any polytope edge have neighboring Voronoi regions.

Zaks [29] described a general procedure to modify any neighborly family of unbounded polyhedra of any dimension, where each polyhedron contains an unbounded circular cone, so that the resulting polytopes are symmetric about a flat of any prescribed dimension.

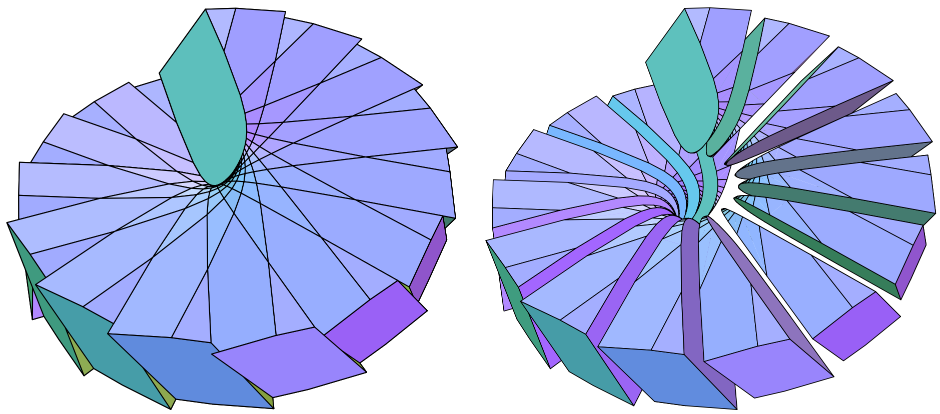
Danzer, Grünbaum, and Klee [9] asked if there is a largest neighborly family of *congruent* polytopes. Zaks (with Linhart) [29] observed that Klee's Voronoi diagram of evenly distributed points on the trigonometric moment curve forms a neighborly family of congruent convex polyhedra in even dimensions four and higher, but left the three-dimensional case open. The largest previously published neighborly family of congruent 3-polytopes, discovered by Zaks [30], consists of eight triangular prisms. According to Croft, Falconer, and Guy [8, Problem E7], this was also the largest known collection of congruent convex bodies in $\mathbb{R}^3$ with the property that every pair has even one (distinct) point of contact. Both Zaks [30] and Croft, Falconer, and Guy [8] conjectured that the largest neighborly family of congruent 3-polytopes is finite (see also Moser and Pach [20, Problem 55]).

In Section 2, we show that this conjecture is incorrect, by giving a constructive proof of the following theorem.

[2] The number of real roots of a polynomial is no more than the number of sign changes in its degree-ordered sequence of non-zero coefficients.

Main Theorem. *For any positive integer n, there is a neighborly family of n congruent convex 3-polytopes.*

Like the earlier constructions of Dewdney and Vranch [10] and Zaks and Linhart [30], our construction is based on the Voronoi diagram of a set of points on a curve, namely the regular circular helix $h(t) = (t, \cos t, \sin t)$. An example of our construction is shown in Figure 1, and a single polytope in our family is shown in Figure 3.



(a) (b)

FIGURE 1. (a) A neighborly family of sixteen congruent convex polytopes. (b) An exploded view of the same family.

Our neighborly family (or a linear transformation thereof) was discovered in the late 1980s by the second author, who was inspired by the way playing cards overlap when they are fanned. However, except for a brief announcement by Gardner [14] (which was unnoticed by most of the mathematics community), the construction was never published. The same construction was independently discovered by the first author in 2001, as a result of his research on the complexity of three-dimesnional Voronoi diagrams [12].

In Section 3, we generalize our Main Theorem to higher dimensions, by constructing an arbitrarily large family of congruent convex polytopes in $\mathbb{R}^d$, any $\lceil d/2 \rceil$ of which share a unique common boundary face. We also introduce a new family of cyclic polytopes, generalizing both the classic cyclic polytopes and the Petrie polytopes.

2. The Main Theorem

Our construction relies on the following observation, independently discovered by Bochiş and Santos [6, Lemma 4.2] (generalizing their earlier proof of a special case [5, Lemma 2.8]) and the author [12, Lemma 2.1]. We include the proof here for the sake of completeness.

Lemma 1. *Let $\beta(t)$ denote the unique sphere passing through $h(t)$ and $h(-t)$ and tangent to the helix at those two points. For any $0 < t < \pi$, the sphere $\beta(t)$ intersects the helix only at its two points of tangency.*

Proof. Since a 180° rotation about the y-axis maps $h(t)$ to $h(-t)$ and leaves the helix invariant, the bitangent sphere $\beta(t)$ must be centered on the y-axis. Thus, $\beta(t)$ is described by the equation $x^2 + (y - a)^2 + z^2 = r^2$ for some constants a and r. Let γ denote the intersection curve of $\beta(t)$ and the cylinder $y^2 + z^2 = 1$. Every intersection point between $\beta(t)$ and the helix must lie on this curve. If we project the helix h and the intersection curve γ to the xy-plane, we obtain the sinusoid $y = \cos x$ and a portion of the parabola $y = \gamma(x) = (x^2 - r^2 + a^2 + 1)/2a$. These two curves meet tangentially at the points $(t, \cos t)$ and $(-t, \cos t)$. See Figure 2.

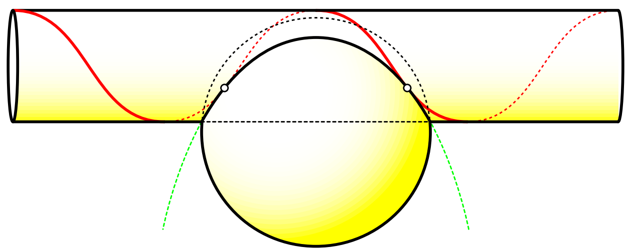


FIGURE 2. The intersection curve of the cylinder and a bitangent sphere projects to a parabola on the xy-plane.

The mean value theorem implies that the equation $\gamma(x) = \cos x$ has at most four solutions in the range $-\pi < x < \pi$. (Otherwise, the curves $y'' = -\cos x$ and $y'' = \gamma''(x) = 1/a$ would intersect more than twice in that range.) Since the curves meet with even multiplicity at two points, those are the only intersection points in the range $-\pi < x < \pi$. Since $\gamma(x)$ is concave, we have $\gamma(\pm\pi) < \cos \pm\pi = -1$, so there are no intersections with $|x| \geq \pi$. Thus, the curves meet only at their two points of tangency. □

Lemma 1 immediately implies that the Voronoi diagram of any finite set of points on the helix $(t, \cos t, \sin t)$ in the range $-\pi < t < \pi$ is a neighborly collection of unbounded convex polyhedra.

To obtain a neighborly family of *congruent* polyhedra, we use the Voronoi diagram of evenly spaced points on the helix. For any positive integer n, let $h_n(t) = h(2\pi t/n)$ and let $\mathcal{H}_n$ denote the infinite point set $\{h_n(t) \mid t \in \mathbb{Z}\}$. By Lemma 1, the Voronoi regions of any $n+1$ consecutive points in $\mathcal{H}_n$ form a neighborly family of convex bodies. Since the point set $\mathcal{H}_n$ is preserved by the rigid motion

$$(x, y, z) \mapsto \left(x + \frac{2\pi}{n},\ y \cos\frac{2\pi}{n} - z \sin\frac{2\pi}{n},\ y \sin\frac{2\pi}{n} + z \cos\frac{2\pi}{n}\right),$$

which maps each point $h_n(t)$ to its successor $h_n(t+1)$, these Voronoi regions are all congruent.

The following more refined analysis of the Delaunay triangulation of $\mathcal{H}_n$, reminiscent of Gale's 'evenness condition' for cyclic polytopes [13, 24], implies that these Voronoi regions have only a finite number of facets, and thus are actually polyhedra.

Lemma 2. *For any integers $a < b < c < d$, the points $h_n(a), h_n(b), h_n(c), h_n(d)$ are vertices of a simplex in the Delaunay triangulation of $\mathcal{H}_n$ if and only if $b - a = d - c = 1$ and $d - a \leq n$.*

Proof. Call a tetrahedron with vertices $h_n(a), h_n(b), h_n(c), h_n(d)$ *local* if $b - a = d - c = 1$ and $d - a \leq n$. Let σ be the sphere passing through the vertices of an arbitrary local tetrahedron. Analysis similar to the proof of Lemma 1 implies that the only portions of the helix that lie inside σ are the segments between $h_n(a)$ and $h_n(b)$ and between $h_n(c)$ and $h_n(d)$. Thus, all other points in $\mathcal{H}_n$ lie outside σ, so the four points form a Delaunay simplex.

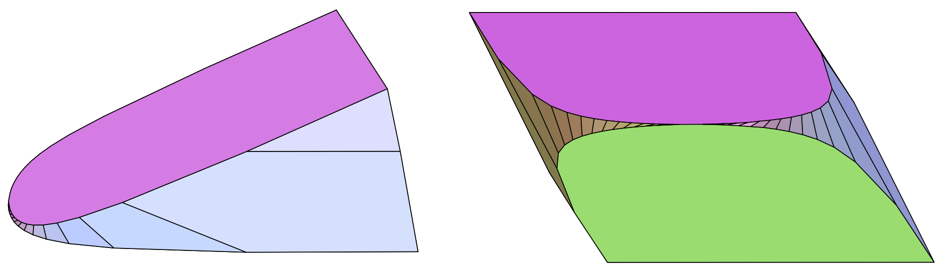
The local Delaunay simplices exactly fill the convex hull of $\mathcal{H}_n$, and therefore comprise the entire Delaunay triangulation. Specifically, the only triangles that are facets of exactly one local tetrahedron have vertices $h_n(i)$, $h_n(i+1), h_n(i+n)$ or $h_n(i-n+1), h_n(i), h_n(i+1)$ for some integer i. Thus, a tetrahedron is Delaunay if and only if it is local. □

In light of the duality between Delaunay triangulations and Voronoi diagrams, Lemma 2 lets us exactly describe the combinatorial structure of the Voronoi regions of $\mathcal{H}_n$. Let $V_n(t)$ denote the Voronoi region of $h_n(t)$. This polyhedron has exactly $2n$ facets, in n symmetric pairs, as follows:

- two unbounded $(2n-1)$-gons shared with $V_n(t \pm 1)$, each bounded by $2n - 3$ segments and two parallel rays;
- two triangles shared with $V_n(t \pm 2)$;
- $2n - 8$ bounded quadrilaterals shared with $V_n(t \pm 3), V_n(t \pm 4), \ldots, V_n(t \pm (n-2))$;
- two unbounded quadrilaterals shared with $V_n(t \pm (n-1))$, each bounded by two line segments and two parallel rays;
- two wedges in parallel planes shared with $V_n(t \pm n)$, each bounded by a pair of rays.

The two $(2n-1)$-gons are adjacent to all the other facets, including each other, and contain all the vertices of $V_n(t)$; otherwise, the facets are adjacent in sequence. See Figure 3.

The vertex of $V_n(0)$ furthest from the x-axis is the center of the sphere through the points $h_n(0)$, $h_n(1)$, $h_n(n-1)$, and $h_n(n)$; this point has coordinates $(\pi, (2\pi - \theta)\theta/(2 - 2\cos\theta), 0)$, where $\theta = 2\pi/n$. Thus, all the Voronoi vertices of $\mathcal{H}_n$ lie in a cylinder of radius $(2\pi - \theta)\theta/(2 - 2\cos\theta) \approx n - 1$ around



(a) (b)

FIGURE 3. (a) One polytope in the neighborly family of sixteen shown in Figure 1. (b) An orthographic edge-on view of the same polytope.

the x-axis. To transform our neighborly family of unbounded polyhedra into a neighborly family of *polytopes*, we intersect each Voronoi region $V_n(t)$ with the halfspace $y\cos(2\pi t/n) + z\sin(2\pi t/n) \leq n$, which contains some positive area of every facet of $V_n(t)$.

This completes the proof of the Main Theorem.

Zaks [31] describes an alternate proof, based entirely on the neighborliness of the Voronoi regions of $\mathcal{H}_n$. For each integer $1 \leq t \leq n$, place a triangle on the shared boundary facet between $V_n(0)$ and $V_n(t)$. Now place congruent copies of these triangles on the boundary of every Voronoi region, so that the entire collection has the same screw symmetry as $\mathcal{H}_n$. Finally, for any integer t, let $C_n(t)$ be the convex hull of the triangles on the boundary of $V_n(t)$. The $n+1$ congruent convex polytopes $C_n(0), C_n(1), \ldots, C_n(n)$ form a neighborly family.

To actually construct our neighborly family (or Zaks'), it suffices to compute the Voronoi diagram of the finite point set $\{h_n(t) \mid t = 0, 1, \ldots, 3n\}$ and then consider only the Voronoi regions of the middle $n+1$ points $h_n(n), h_n(n+1), \ldots, h_n(2n-1), h_n(2n)$, since those Voronoi regions are the same as in the infinite point set $\mathcal{H}_n$. Figure 1 was computed using this method.

Since a $180°$ rotation about the y-axis maps each point $h_n(t)$ to $h_n(-t)$, and thus preserves the point set $\mathcal{H}_n$, the Voronoi region $V_n(0)$ is rotationally symmetric about the y-axis. It immediately follows every Voronoi region of $\mathcal{H}_n$ has a line of $180°$ rotational symmetry. Clipping each Voronoi region by an additional halfspace as above retains this symmetry, since the clipping plane is normal to the symmetry axis. We can create a neighborly family of congruent polytopes with additional symmetries by taking the union of each clipped Voronoi region and its reflection across its clipping plane. Each resulting polytope clearly has bilateral symmetry about its clipping plane and 180° symmetry about the original Voronoi region's axis of symmetry, and therefore is centrally symmetric about the intersection point of the clipping plane and the symmetry axis.

Theorem 3. *For any integer positive integer n, there is a neighborly family of n congruent convex 3-polytopes, each with a plane of bilateral symmetry, a line of 180° rotational symmetry, and a point of central symmetry.*

3. Higher Dimensions

A family of convex polyhedra in $\mathbb{R}^d$ is *(strictly) k-neighborly* if any subset of k polyhedra has a $(d-k+1)$-dimensional intersection, and no subset of $k+1$ polytopes has a non-empty intersection.[3] Arbitrarily large k-neighborly families of polyhedra are easy to construct in $\mathbb{R}^{2k-1}$, for example, Schlegel diagrams of dual cyclic $2k$-polytopes [7, 13] or Voronoi diagrams of points on the moment curve [25]. However, arbitrarily large k-neighborly families of *congruent* polyhedra were previously only known in dimensions $2k$ and higher. The lowest-dimensional example is based on the Voronoi diagram of evenly distributed points on the trigonometric moment curve [18, 29] together with the origin (since otherwise the origin is on the boundary of every Voronoi polyhedron).

In this section, we generalize our three dimensional results by considering regularly spaced points on the following *generalized helix*:

$$h_k(t) = \big(t,\ \cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \ldots,\ \cos kt,\ \sin kt\big) \in \mathbb{R}^{2k+1}.$$

Theorem 4. *Let P be any finite set of points on the curve $h_k(t)$ in the range $0 < t < 2\pi$, for some non-negative integer k. The Voronoi diagram of P is a $(k+1)$-neighborly family of convex polyhedra in $\mathbb{R}^{2k+1}$.*

Proof. Consider the sphere σ passing through $k+1$ arbitrary points $h_k(a_0)$, $h_k(a_1), \ldots, h_k(a_k) \in P$ and tangent to the generalized helix at those points, where $0 < a_0 < a_1 < \cdots < a_k < 2\pi$. Any point $h_k(t)$ that lies on σ satisfies the following $(2k+3) \times (2k+3)$ matrix equation:

[3] The second condition is necessary to rule out degenerate constructions such as the product of a $(d-2)$-dimensional cube with n congruent planar wedges.

$$F(t) = \begin{vmatrix} 1 & a_0 & \cos a_0 & \sin a_0 & \cdots & \cos ka_0 & \sin ka_0 & k + a_0^2 \\ 0 & 1 & -\sin a_0 & \cos a_0 & \cdots & -k \sin ka_0 & k \cos ka_0 & 2a_0 \\ 1 & a_1 & \cos a_1 & \sin a_1 & \cdots & \cos ka_1 & \sin ka_1 & k + a_1^2 \\ 0 & 1 & -\sin a_1 & \cos a_1 & \cdots & -k \sin ka_1 & k \cos ka_1 & 2a_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & a_k & \cos a_k & \sin a_k & \cdots & \cos ka_k & \sin ka_k & k + a_k^2 \\ 0 & 1 & -\sin a_k & \cos a_k & \cdots & -k \sin ka_k & k \cos ka_k & 2a_k \\ 1 & t & \cos t & \sin t & \cdots & \cos kt & \sin kt & k + t^2 \end{vmatrix} = 0$$

(Each of the even rows of this matrix is the derivative of the preceding row.) To bound the number of zeros of $F(t)$, consider its second derivative

$$F''(t) = \begin{vmatrix} 1 & a_0 & \cos a_0 & \sin a_0 & \cdots & \cos ka_0 & \sin ka_0 & k + a_0^2 \\ 0 & 1 & -\sin a_0 & \cos a_0 & \cdots & -k \sin ka_0 & k \cos ka_0 & 2a_0 \\ 1 & a_1 & \cos a_1 & \sin a_1 & \cdots & \cos ka_1 & \sin ka_1 & k + a_1^2 \\ 0 & 1 & -\sin a_1 & \cos a_1 & \cdots & -k \sin ka_1 & k \cos ka_1 & 2a_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & a_k & \cos a_k & \sin a_k & \cdots & \cos ka_k & \sin ka_k & k + a_k^2 \\ 0 & 1 & -\sin a_k & \cos a_k & \cdots & -k \sin ka_k & k \cos ka_k & 2a_k \\ 0 & 0 & -\cos t & -\sin t & \cdots & -k^2 \cos kt & -k^2 \sin kt & 2 \end{vmatrix}.$$

$F''(t)$ is an affine combination of the functions $\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt$, so it can be rewritten as as a polynomial of degree at most $2k$ in the variable e^{it}. Thus, $F''(t)$ has at most $2k$ zeros in the range $0 < t < 2\pi$. (This is essentially the argument used by Carathéodory to show that Petrie polytopes are neighborly [7].)

Since $a_0, a_1, \ldots, a_k$ are roots of $F(t)$ of multiplicity two, they are the only roots in the range $0 < t < 2\pi$; otherwise, by the mean value theorem, $F''(k)$ would have more than $2k$ roots in the range $0 \le a_0 < t < a_k \le 2\pi$, which we have just shown to be impossible. Thus, the points $h_k(a_0), h_k(a_1), \ldots, h_k(a_k)$ lie on a sphere that excludes every other point in P and so have mutually neighboring Voronoi regions. □

In fact, Theorem 4 is a special case of the following result, which follows from an easy generalization of the previous proof and Gale's evenness condition for cyclic polytopes [13, 24]. Define the *mixed moment curve* $\mu_{d,k}(t)$ as follows:

$$\mu_{d,k}(t) = \bigl(t, t^2, \ldots, t^d,\ \cos t, \sin t,\ \cos 2t, \sin 2t,\ \ldots, \cos kt, \sin kt\bigr) \in \mathbb{R}^{2k+d}.$$

For example, $\mu_{d,0}(t)$ is the standard d-dimensional moment curve, $\mu_{0,k}(t)$ is the $2k$-dimensional trigonometric moment curve, and $\mu_{1,k}(t)$ is our generalized helix.

Theorem 5. *For any non-negative integers d and k, the convex hull of any finite set of points on the curve $\mu_{d,k}(t)$ in the range $0 < t < 2\pi$ is a $(d+2k)$-dimensional cyclic polytope.*

Not surprisingly, we obtain arbitrarily large highly-neighborly families of *congruent* polytopes by considering the Voronoi diagram of the infinite evenly-spaced point set $\mathcal{H}_n^k = \{h_k(2\pi t/n) \mid t \in \mathbb{Z}\}$.

Theorem 6. *For any non-negative integers n and k, any $n+1$ consecutive Voronoi regions in the Voronoi diagram of $\mathcal{H}_n^k$ form a $(k+1)$-neighborly family of congruent convex polyhedra.*

Proof. Fix an integer n. To simplify notation, let $\hbar(t) = h_k(2\pi t/n)$, and let $\langle t_1, t_2, \ldots, t_r\rangle$ denote the convex hull of the points $\hbar(t_1), \hbar(t_2), \ldots, \hbar(t_r)$. Since the set $\mathcal{H}_n^k = \{\hbar(t) \mid t \in \mathbb{Z}\}$ is preserved under a rigid motion mapping each point $\hbar(t)$ to its successor $\hbar(t+1)$, the Voronoi regions of $\mathcal{H}_n^k$ are all congruent.

Call a full-dimensional simplex with vertices in $\mathcal{H}_n^k$ *local* if all its vertices consist of $k+1$ adjacent pairs within a single turn of the generalized helix; in other words, every local simplex has the form

$$\langle a_0,\ a_0+1,\ a_1,\ a_1+1,\ \ldots,\ a_k,\ a_k+1\rangle$$

for some integers $a_0, a_1, \ldots, a_k$ with $a_k+1 \le a_0+n$ and $a_i+1 < a_{i+1}$ for all i. Analysis similar to Theorem 4 implies that every local simplex is Delaunay.

The convex hull of $\mathcal{H}_n^k$, which we will call the *Petrie cylinder*, is the product of an $2k$-dimensional Petrie polytope with n vertices and a line orthogonal to that polytope's hyperplane. By Gale's evenness condition [13, 24], the facets of the Petrie polytope are formed by all sets of k adjacent pairs of points on the trigonometric moment curve. The faces of the Petrie cylinder are cylinders over the faces of the Petrie polytope.

Call a facet of a local simplex that is not shared by another local simplex a *boundary simplex.* We easily observe that the boundary simplices are exactly the $2k$-simplices with one of the following two forms:

$$\langle a_k-n+1,\ a_1,\ a_1+1,\ a_2,\ a_2+1,\ \ldots, a_k,\ a_k+1\rangle$$
$$\langle a_1,\ a_1+1,\ a_2,\ a_2+1,\ \ldots, a_k,\ a_k+1,\ a_1+n\rangle$$

The following infinite sequence of boundary simplices exactly covers one facet of the Petrie cylinder.

$$\begin{array}{l} \ddots \\ \langle a_k - n + 1, a_1, a_1 + 1, a_2, a_2 + 1, \ldots, a_k, a_k + 1\rangle \\ \quad \langle a_1, a_1 + 1, a_2, a_2 + 1, \ldots, a_k, a_k + 1, a_1 + n\rangle \\ \quad\quad \langle a_1 + 1, a_2, a_2 + 1, \ldots, a_k, a_k + 1, a_1 + n, a_1 + n + 1\rangle \\ \quad\quad\quad \langle a_2, a_2 + 1, \ldots, a_k, a_k + 1, a_1 + n, a_1 + n + 1, a_2 + n\rangle \\ \quad\quad\quad\quad \ddots \end{array}$$

Every facet of the Petrie cylinder is covered in this manner, and every boundary simplex lies on some facet of the Petrie cylinder. Thus, the union of the boundary facets is the boundary of the Petrie cylinder, so the local Delaunay simplices completely fill the Petrie cylinder and therefore comprise the entire Delaunay triangulation.

It easily follows that each Voronoi region of $\mathcal{H}_n^k$ is a convex polyhedron with $\Theta(n^k)$ facets, and that any $n+1$ consecutive Voronoi regions form a $(k+1)$-neighborly family. As we already observed, these polyhedra are congruent. □

We can easily modify our construction to obtain a $(k+1)$-neighborly family of polytopes, by intersecting each Voronoi region with a halfspace strictly containing all its vertices. Each Voronoi region of $\mathcal{H}_n^k$ has a k-flat of two-fold symmetry. As long as the boundary of the new halfspace is perpendicular to this central k-flat, the resulting polytope is also symmetric about this flat.

Using a variant of Zaks' symmetrization procedure [29], we can ensure that each polytope is also symmetric about a flat of any specified dimension. Consider the Voronoi region V of $\hbar(0)$ in the Voronoi diagram of $\mathcal{H}_n^k$. Let ρ be the ray from the origin through $\hbar(0)$, let ϕ^+ and ϕ^- denote the supporting hyperplanes of the only two parallel facets of V (shared with the Voronoi regions of $\hbar(n)$ and $\hbar(-n)$), and let π be a hyperplane normal to ρ at sufficient distance from the origin. Finally, let f be any flat that lies in π, contains the point $\rho \cap \pi$, and is either parallel or perpendicular to both ϕ^+ and ϕ^-. The intersection of V and its reflection across f is a convex polytope that is obviously symmetric about f and whose boundary contains positive measure from every boundary facet of V. Applying this procedure to any $n+1$ consecutive Voronoi regions of $\mathcal{H}_n^k$, we obtain our final result.

Theorem 7. *For any non-negative integers k, n, and $r \le 2k$, there is a $(k+1)$-neighborly family of $n+1$ congruent convex polytopes in $\mathbb{R}^{2k+1}$, each of which is symmetric about an r-flat.*

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E-mail address: jeffe@cs.uiuc.edu
E-mail address: scottkim@aol.com