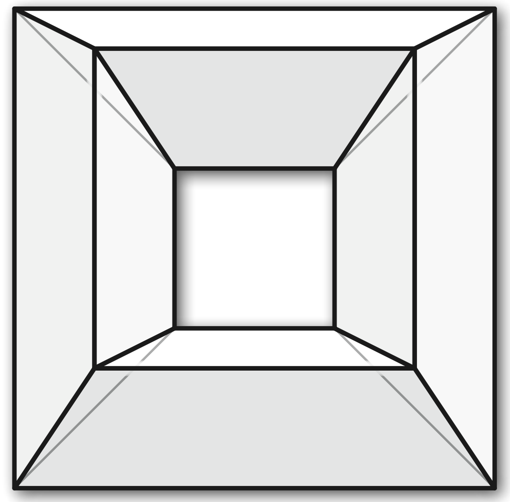 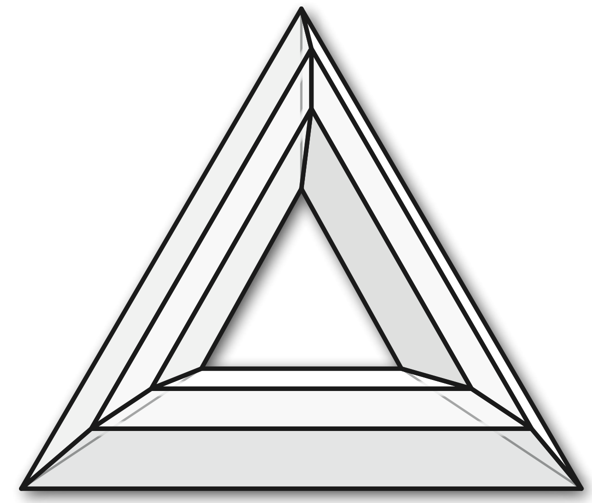

Efficiently Hex-Meshing Things with Topology

Jeff Erickson

University of Illinois, Urbana-Champaign

1st Mathematical Congress of the Americas

Guanajuato, México

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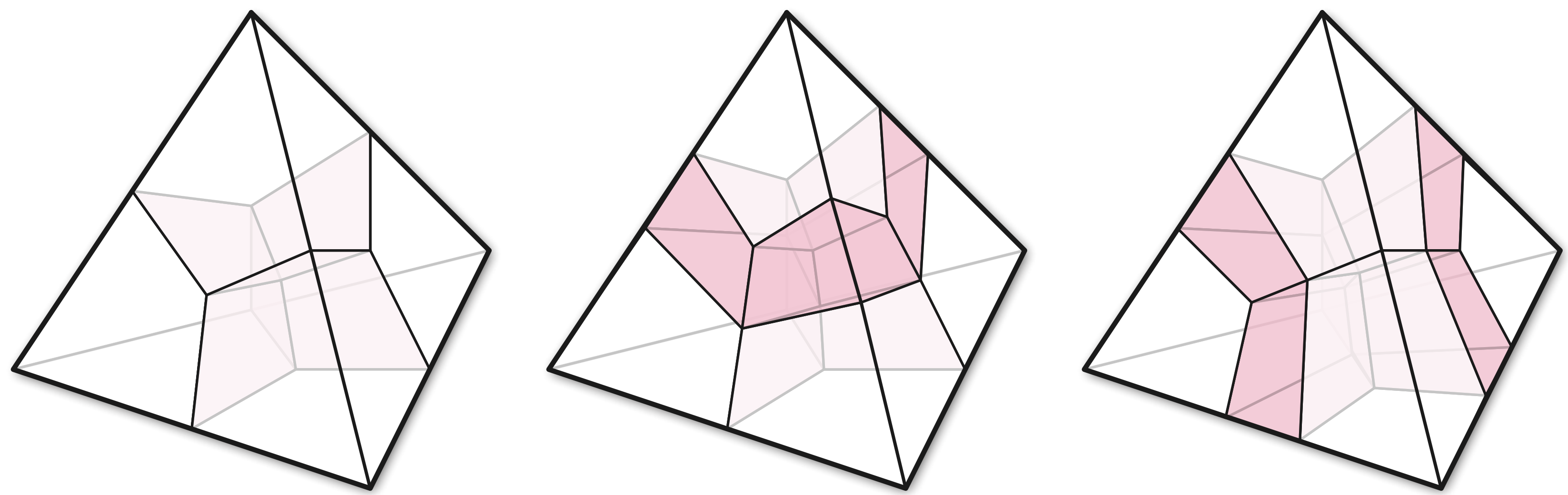

Efficiently Hex-Meshing Things with Topology

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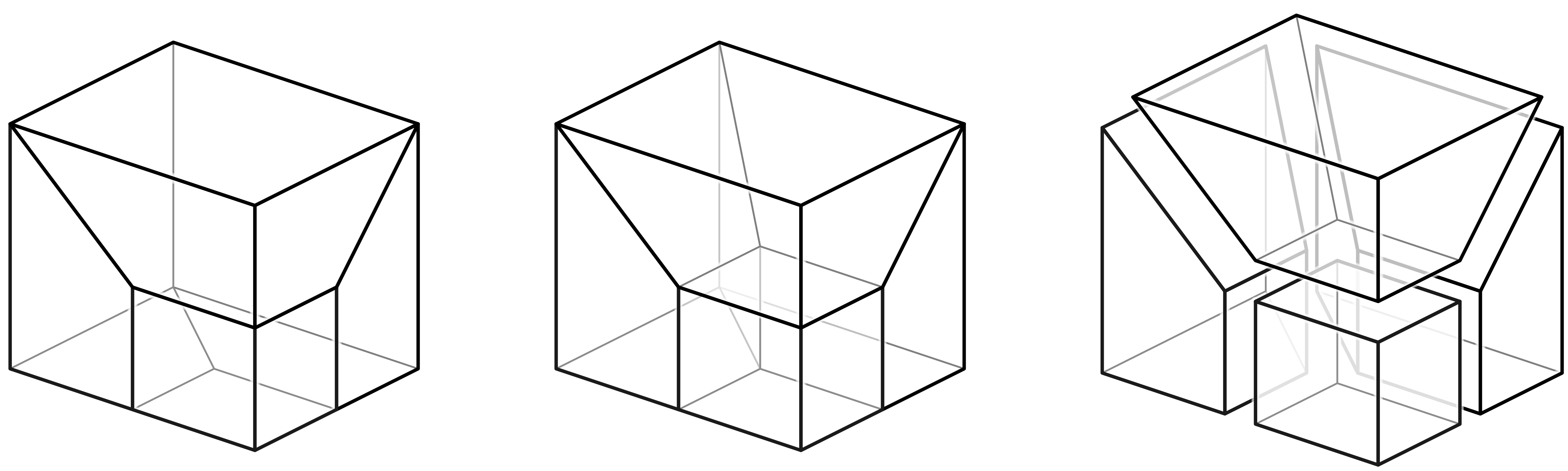
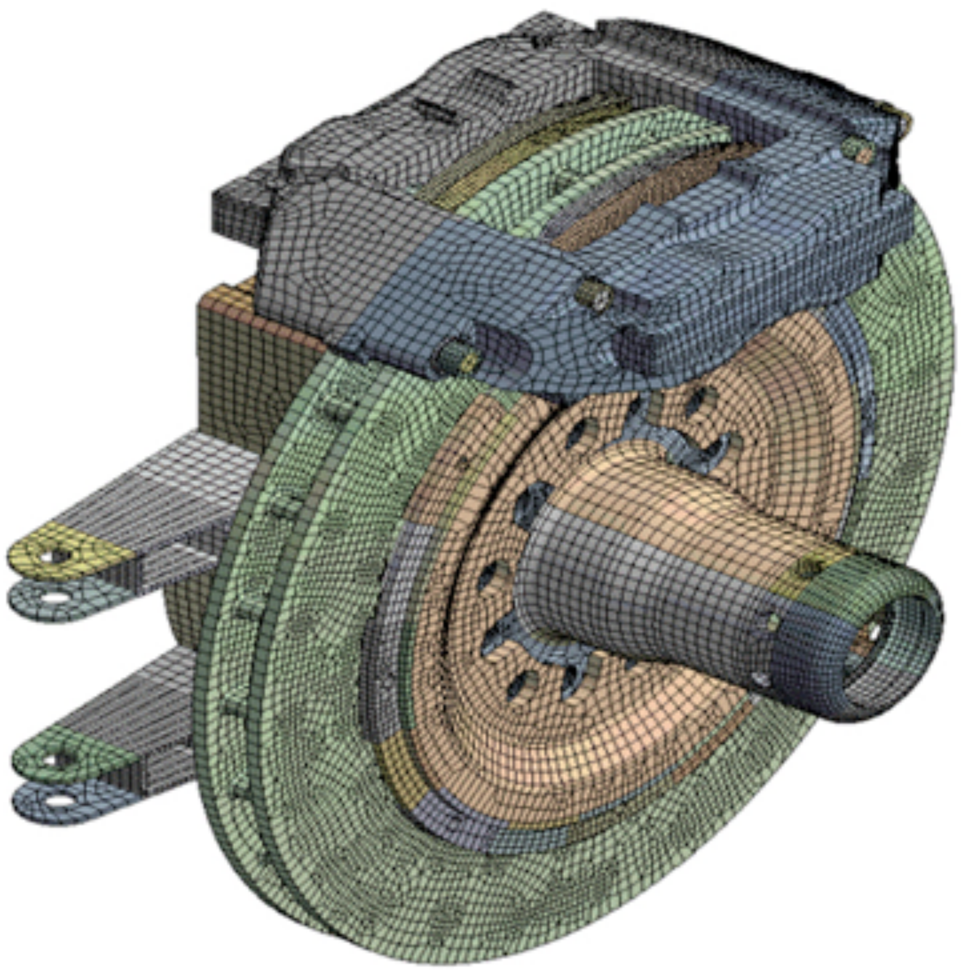
Hex meshing

- **Input:** A polyhedron P in $\mathbb{R}^3$ with quadrilateral facets.
- **Output:** A *hexahedral mesh* of the interior of P whose boundary facets are precisely the facets of P.
(No boundary refinement)

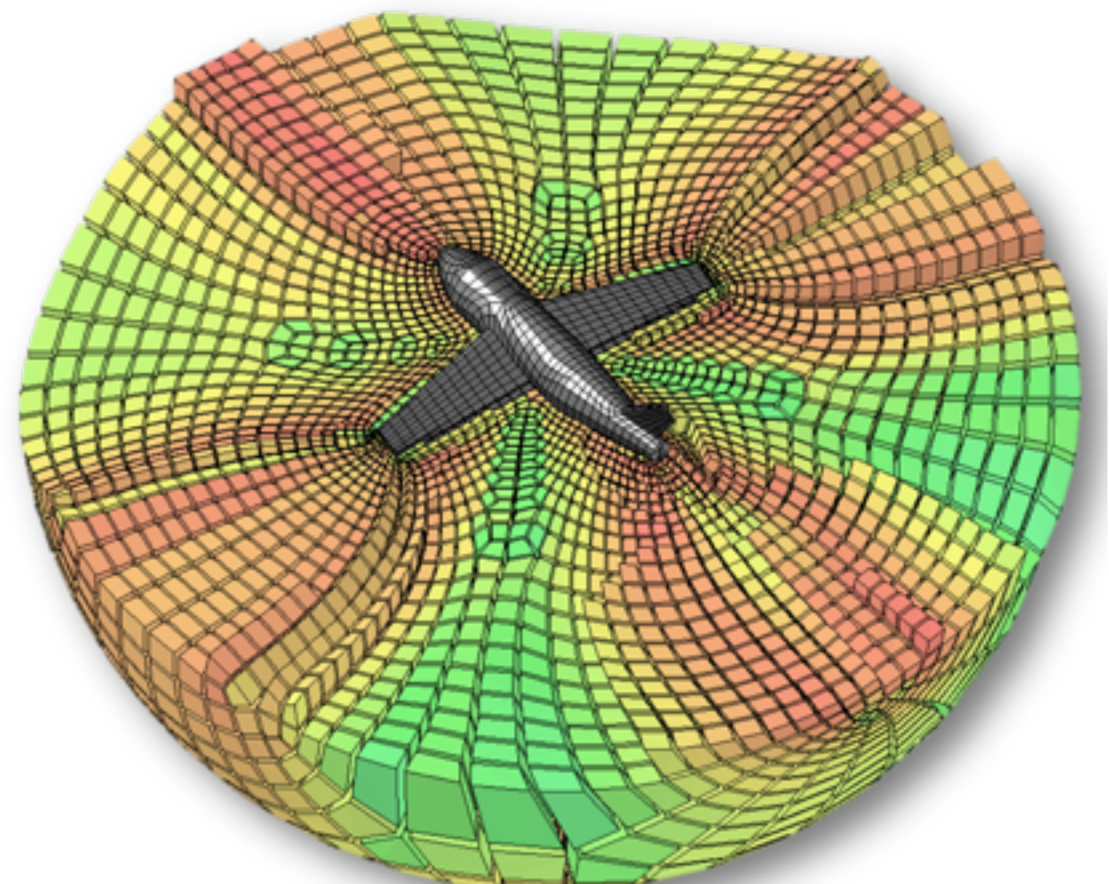


Why hex meshes?

- Numerical solution of PDEs via finite-element methods



[Ansys]



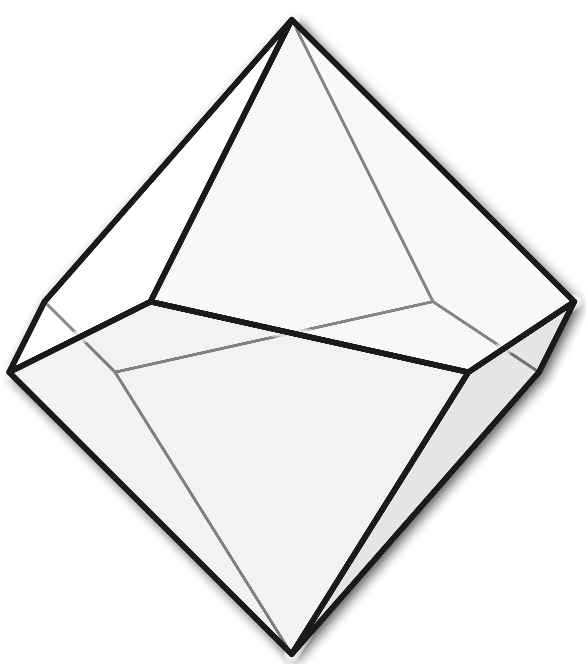
[Ruiz-Gironès Roca Sarrate 2012]

What's a hex mesh?

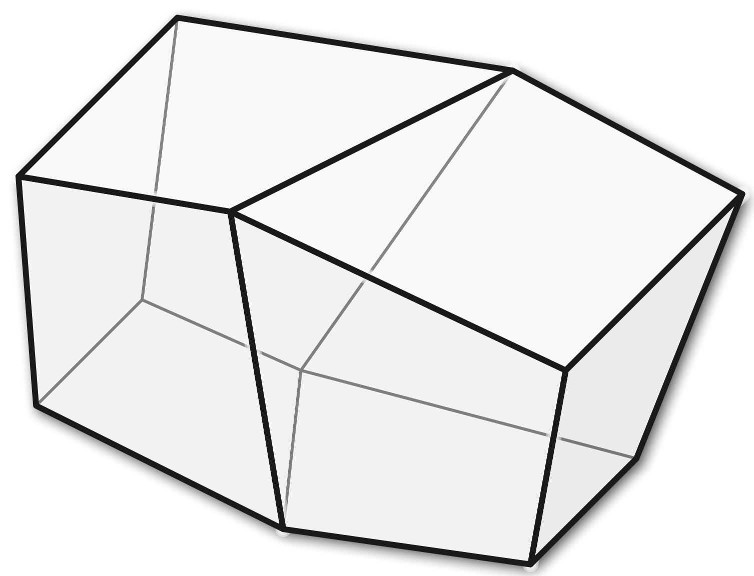
- Geometric version: A cell complex of *convex polyhedra*, each with six quadrilateral facets.

What’s a hex mesh?

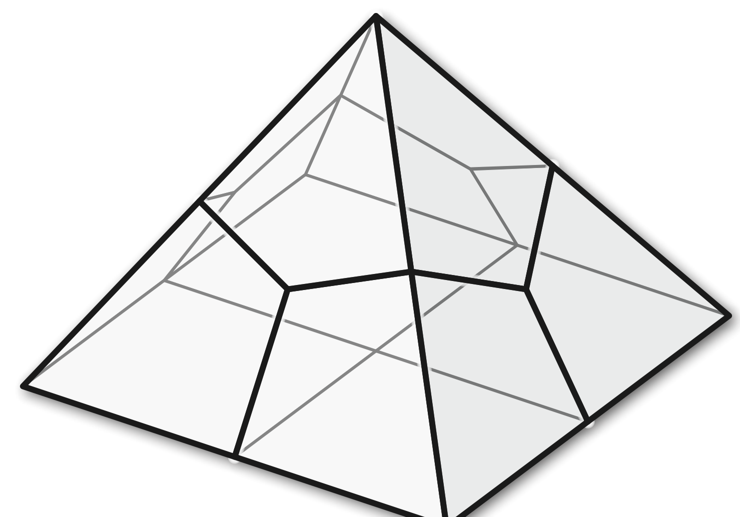
- Geometric version: A cell complex of *convex polyhedra*, each with six quadrilateral facets.
- Unfortunately, the *existence* of polyhedral hex meshes is open even for the simplest nontrivial inputs.



Octagonal spindle



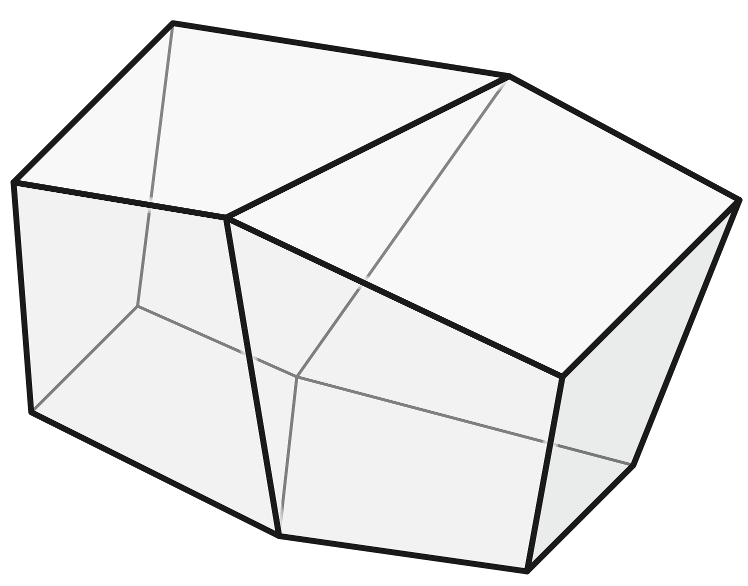
Bicuboid

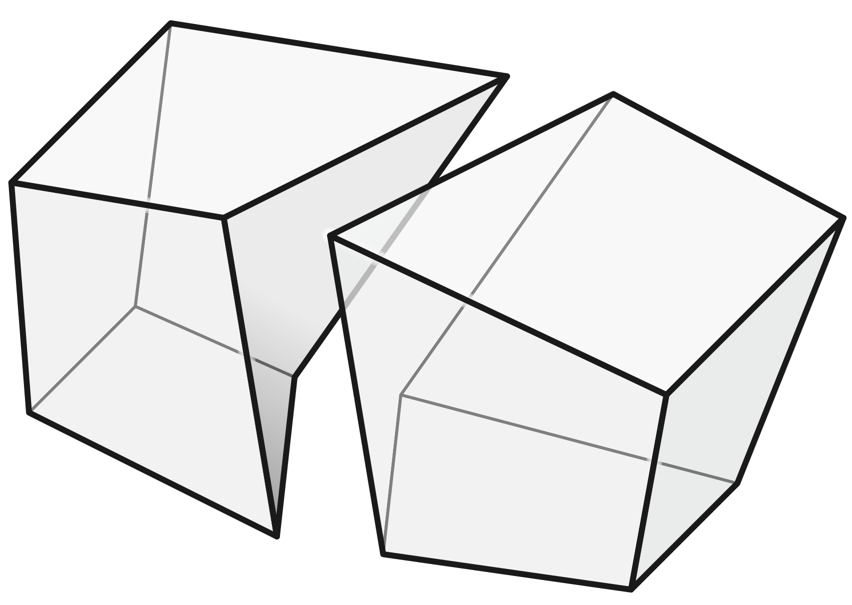


Schneiders’ pyramid

What’s a hex mesh?

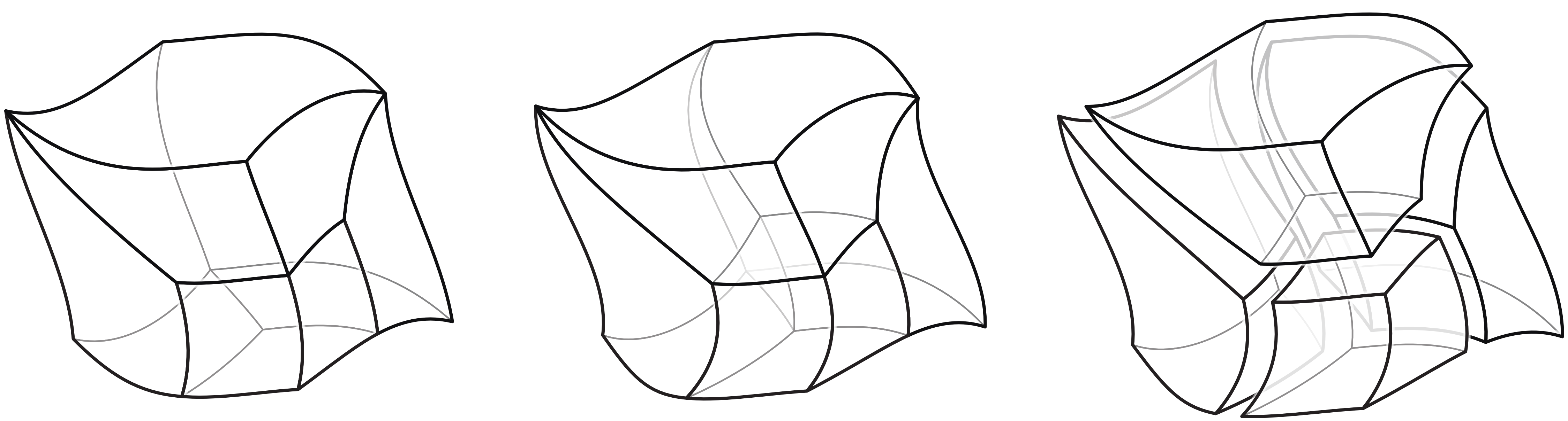
- So instead we look for a standard *topological* cube complex.
- Each cube is a topological ball with the graph of the standard cube drawn on its boundary.
- The intersection of two cubes is either a common facet, a common edge, a common vertex, or empty.





Topological hex meshing

- **Input:** A *topological* quad mesh Q of the boundary of a compact subset Ω of $\mathbf{R}^3$.
- **Output:** A *topological* hex mesh of Ω whose boundary is precisely the quad mesh Q.



Necessary condition

Every hex mesh has an even number of boundary facets.

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- Facets are glued together in pairs.

Necessary condition

Every hex mesh has an even number of boundary facets.

- Each cube has six facets.
- Six is even.
- Facets are glued together in pairs.
- Two is even.

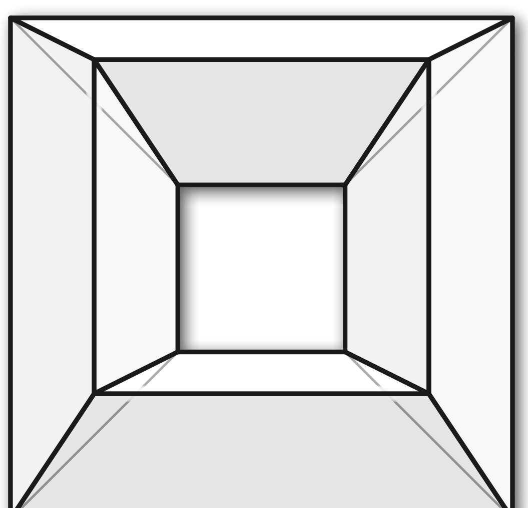
Genus-zero surfaces

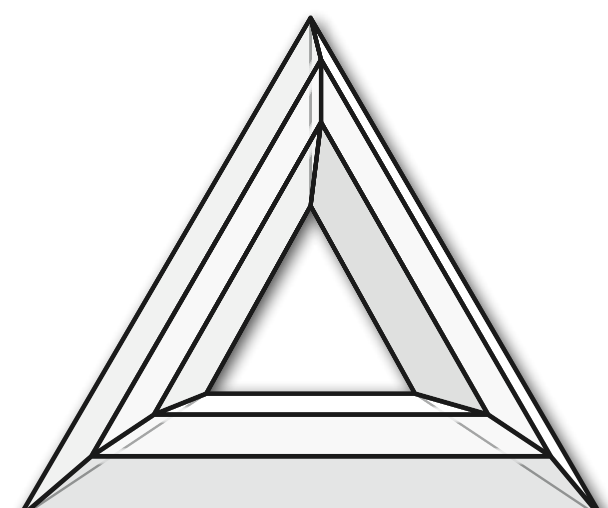
- A quad mesh of *the sphere* is the boundary of a hex mesh of *the ball* if and only if the number of quads is even.
  [Thurston 93, Mitchell 96]

- Every quad mesh of the sphere with $2n$ quads is the boundary of a hex mesh of the ball *with $O(n)$ hexes*.
  [Eppstein 99]

Higher-genus surfaces

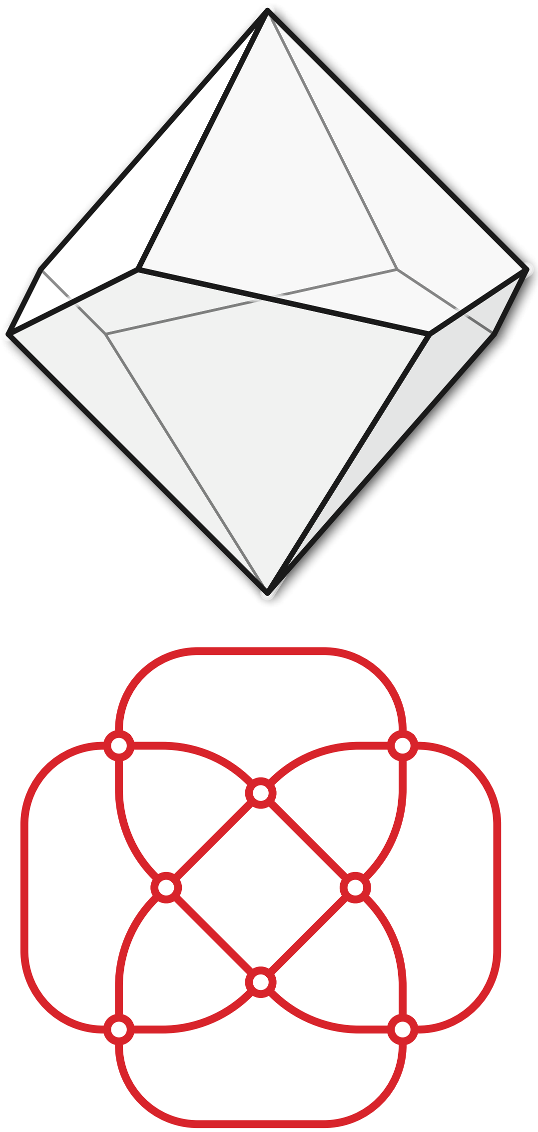
- **Necessary:** No *odd* cycle in Q bounds a *disk* in Ω. [Mitchell 96]
- **Sufficient:** There are g *even* cycles in Q that bound *disks* in Ω that cut Ω into a ball. [Mitchell 96]
- **Sufficient:** Q is bipartite. [Eppstein 99]


No hex mesh

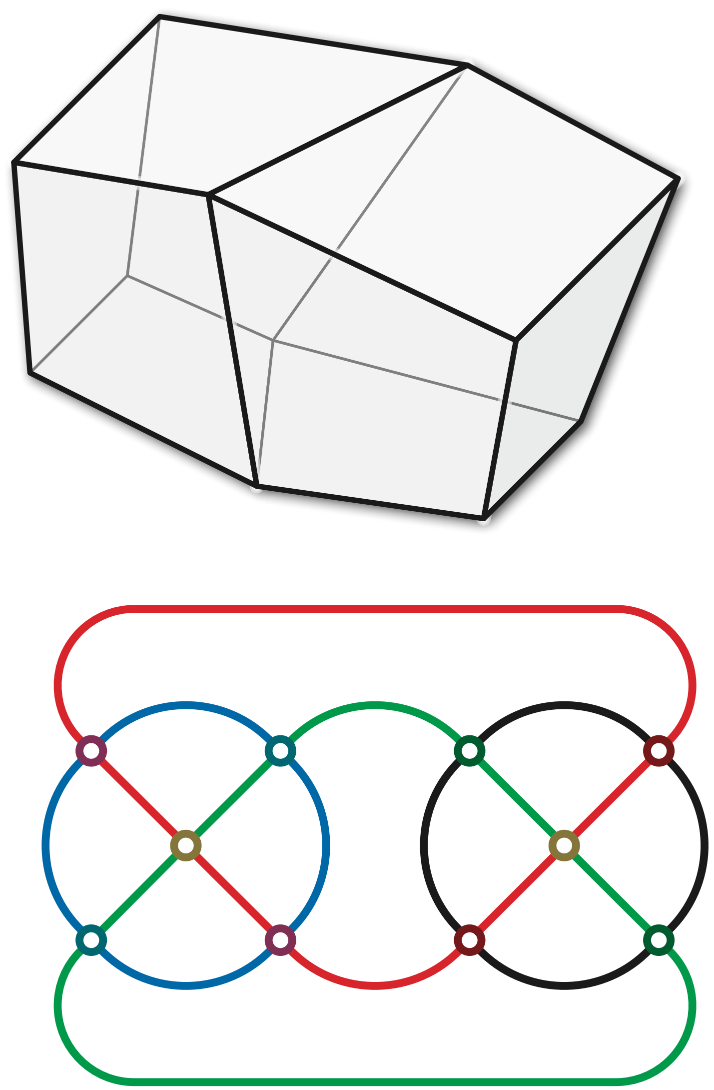

Mesh with 3 cubes

Dual curves

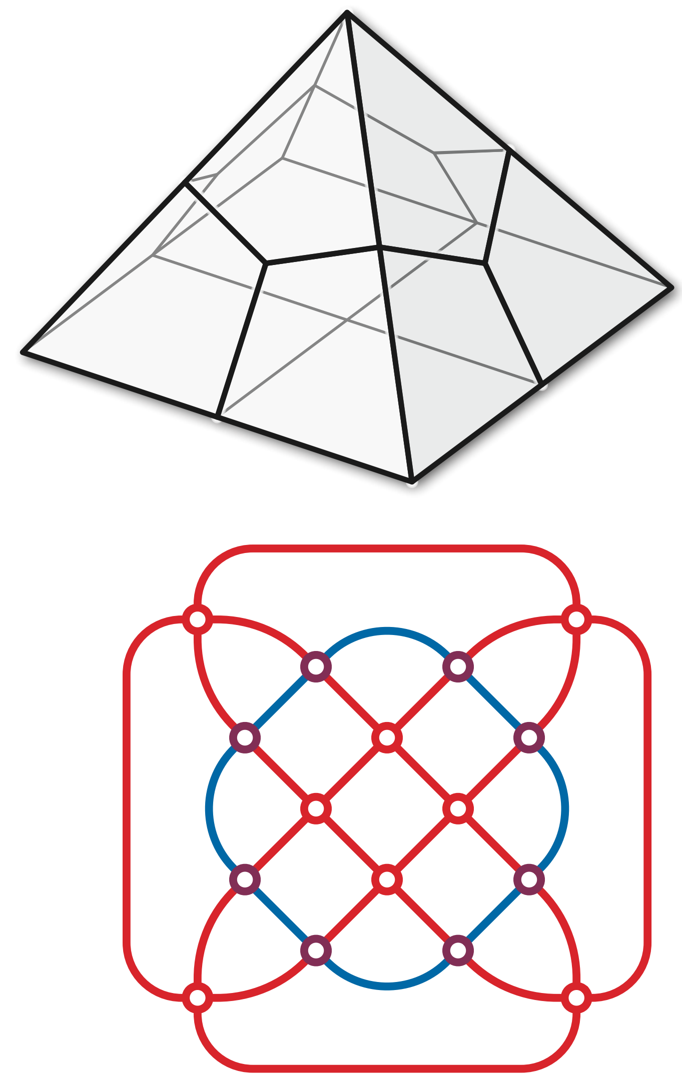
- The dual Q^* of a quad mesh Q is an immersion of circles.



Octagonal spindle



Bicuboid



Schneiders' pyramid

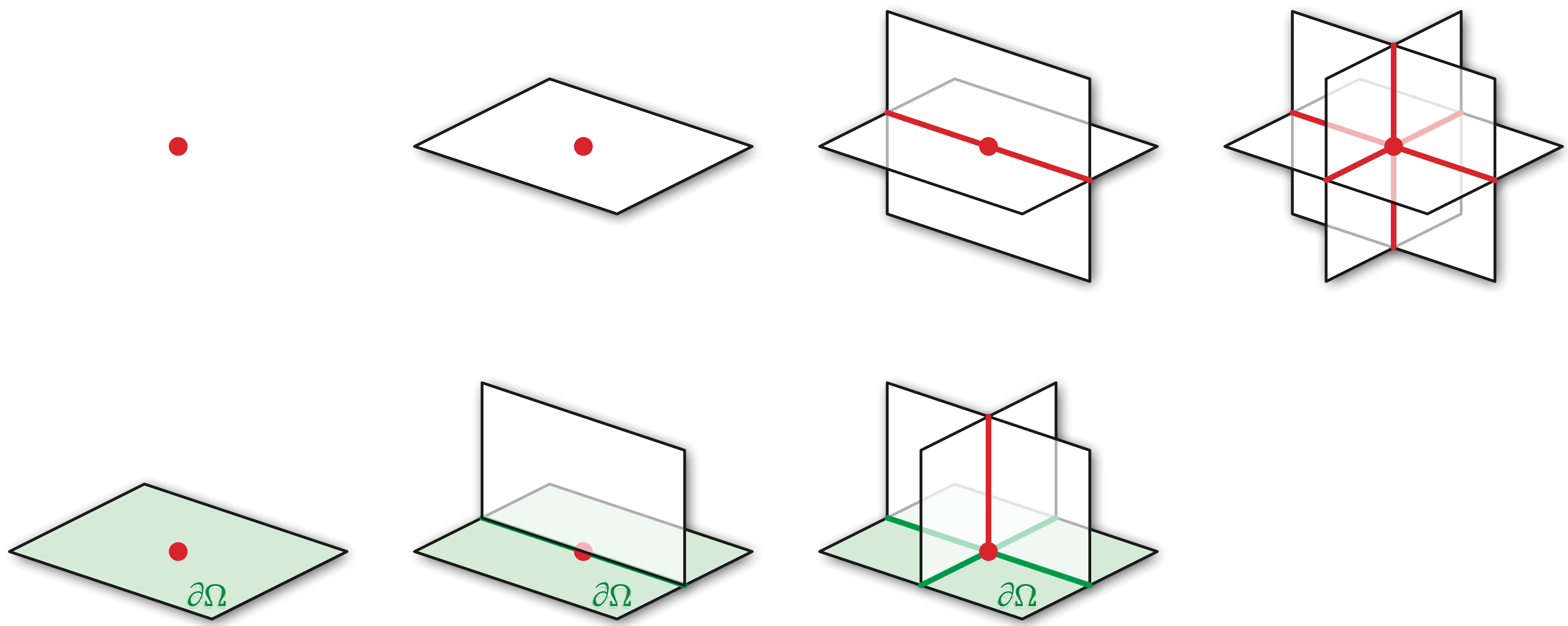
Main result

Let Ω be a compact subset of $\mathbf{R}^3$ whose boundary is a 2-manifold. Let Q be a topological quad mesh of ∂Ω with an even number of facets. The following are equivalent:

- Q is the boundary of a hex mesh of Ω.
- No odd cycle in Q bounds *an immersed surface* in Ω.
- Q^* is the boundary of *an immersed surface* in Ω.

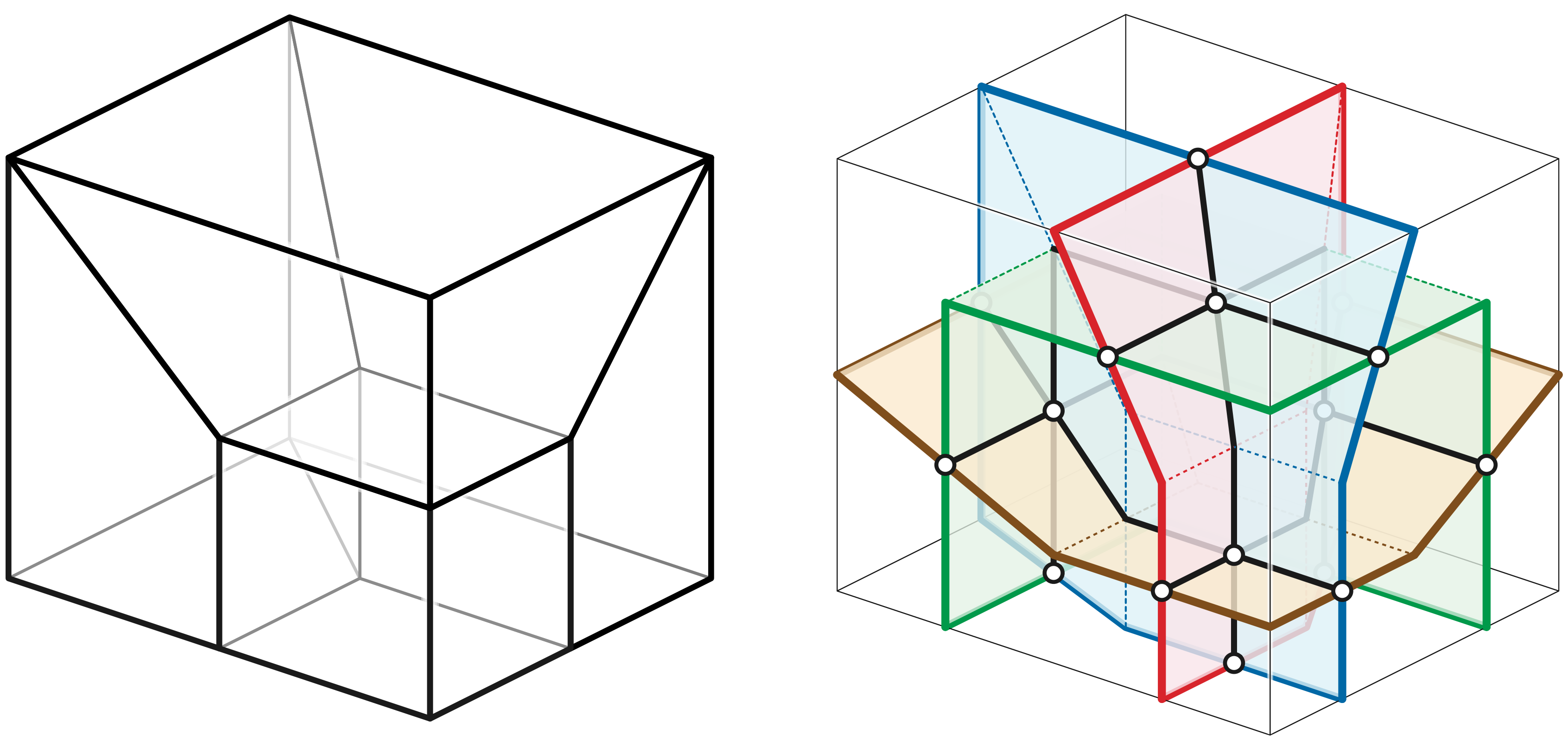
Surface immersion

- Every point in Ω has one of these neighborhoods:



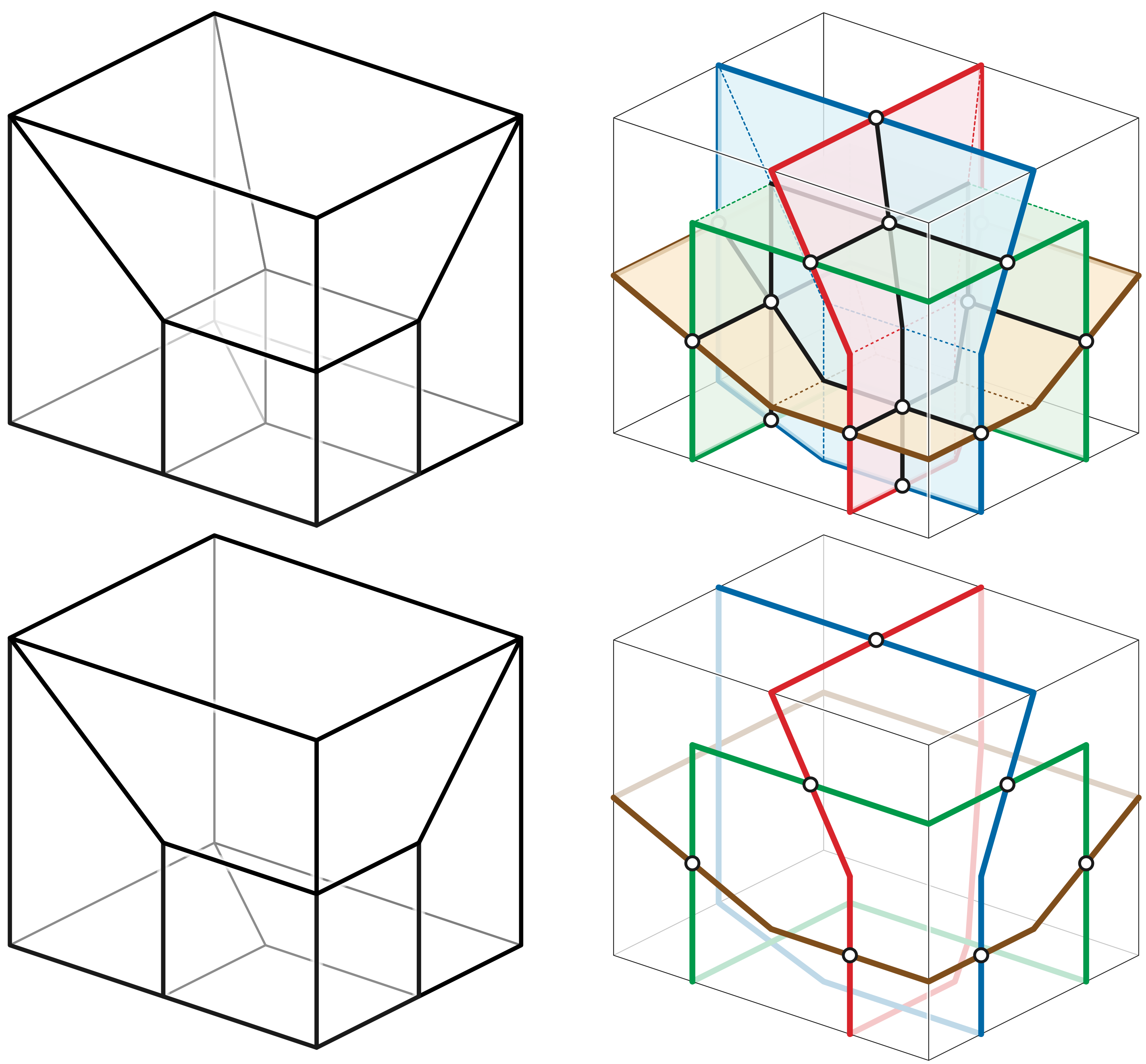
Dual surfaces

- The dual of any hex mesh is a surface immersion.
  - duals of zonotopes [Fedorov 1885, ...]
  - "derivative complex" [Jockusch 93 (MacPherson, Stanley), ...]
  - "hyperplanes" [Sageev 95, ...]
  - "spatial twist continuum" [Murdoch Benzley 95, Mitchell 96, ...]
  - "canonical surface" [Aitchison *et al.* 97, ...]



Dual surfaces

- The boundary of the dual is the dual of the boundary.



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Main result with homology

Let Ω be a compact subset of $\boldsymbol{R}^3$ whose boundary is a 2-manifold. Let Q be a topological quad mesh of ∂Ω with an even number of facets. The following are equivalent:

- Q is the boundary of a hex mesh of Ω.
- No odd cycle in Q is *null-homologous* in Ω.
- Q^* is *null-homologous* in Ω.

(All homology with $\boldsymbol{Z}_2$ coefficients)

Main result with homology

Let Ω be a compact subset of $\boldsymbol{R}^3$ whose boundary is a 2-manifold. Let Q be a topological quad mesh of ∂Ω with an even number of facets. The following are equivalent:

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- Q^* is *null-homologous* in Ω.

Poincaré-Alexander-Lefschetz duality

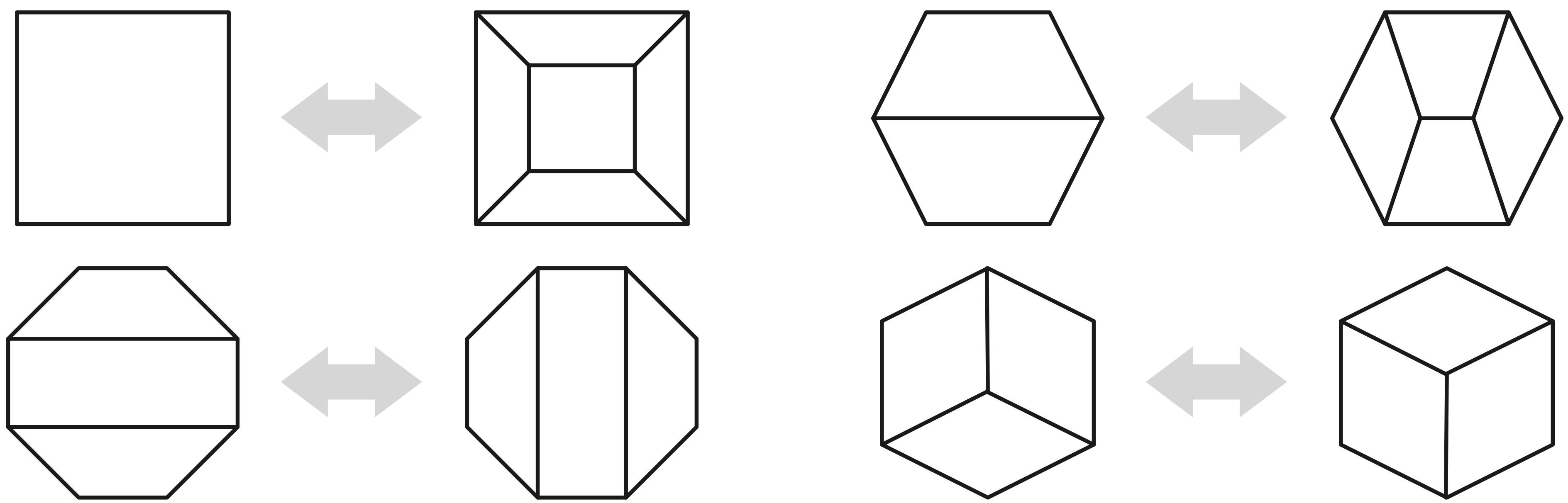
(All homology with $\boldsymbol{Z}_2$ coefficients)

Main result

- Q^* is always null-homologous if Ω is a ball.
- Odd cycle condition is trivial for bipartite meshes.
- The volume Ω need not be a handlebody.
- The boundary $\partial\Omega$ need not be connected.

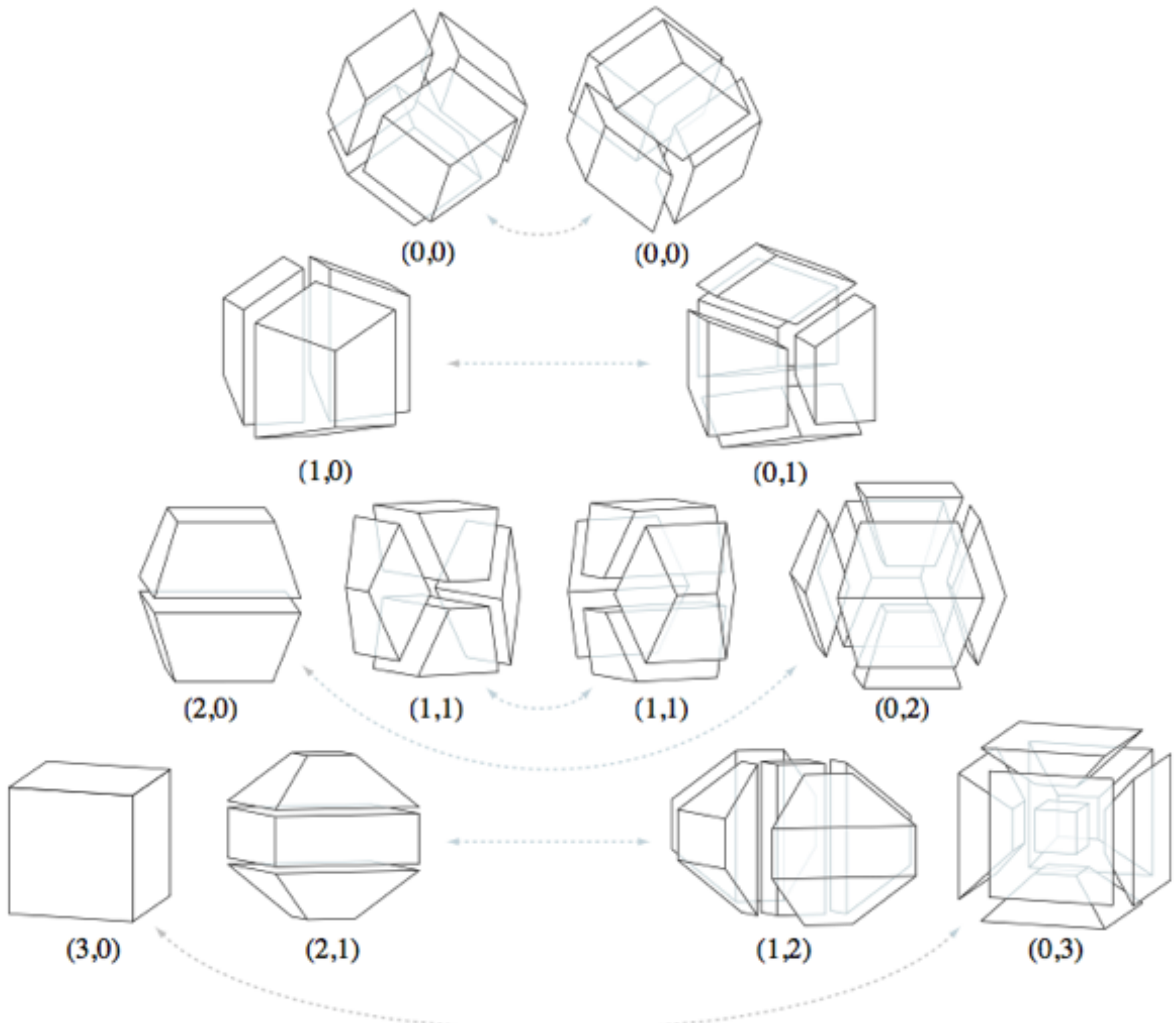
Cube flips

- Replace a simply-connected subset of cube facets with its (simply-connected) complement.



- Analogue of *bistellar flips* for simplicial complexes.
  [Alexander '30, Pachner '78]

Cube flips



[Bern Eppstein '01]

Habegger's Problem

- *Any* two PL triangulations of the same PL manifold are connected by bistellar flips. [Pachner '78, '90]

- When are two PL cubulations of the same PL manifold connected by cube flips? [Problem 5.13 in Kirby's Problems in Low-Dimensional Topology, 1995]

- **Conjecture:** If and only their dual surface immersions are *cobordant*. [Funar '99]

Funar’s theorem

- **Theorem:** Two quad meshes of the same 2-manifold (possibly with boundary) are equivalent if and only if they have the same number of quads mod 2 and homologous dual curves. [Funar ’07]

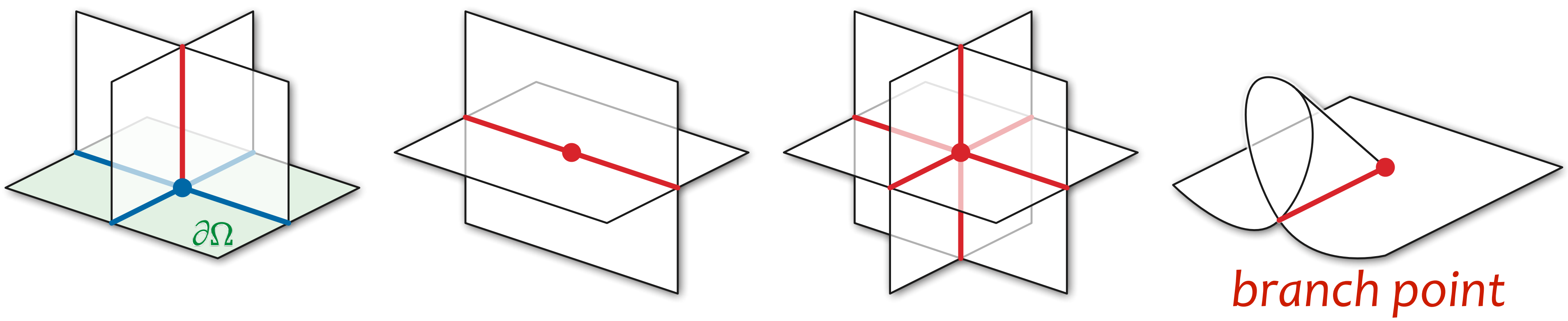
- Habegger’s question is still open for dimension ≥ 3.

Let’s do math!

- Common strategy for Thurston and Mitchell’s genus-zero theorem, Funar’s cube-flip theorem, and our main result
- Suppose Q^* is null homologous in Ω.
- **Step 1:** Q^* is the boundary of a surface immersion S.
- **Step 2:** S can be refined to the dual of a hex mesh.

Extending curves to surfaces

- Since Q^* is null-homologous in Ω, there is a *generic surface map* $f{:}S\to\Omega$ such that $f(\partial S)=Q^*$. [Whitney 44, Papakyriakopoulos 57]



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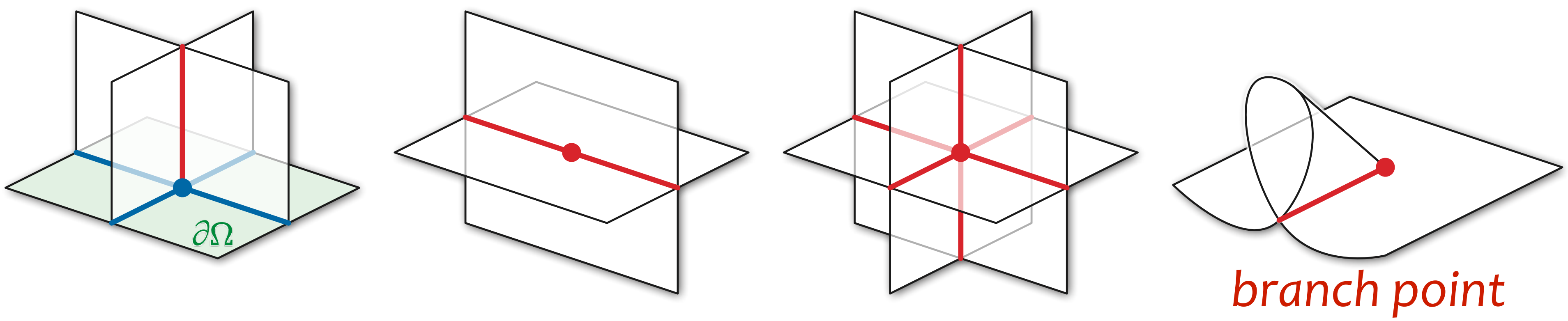
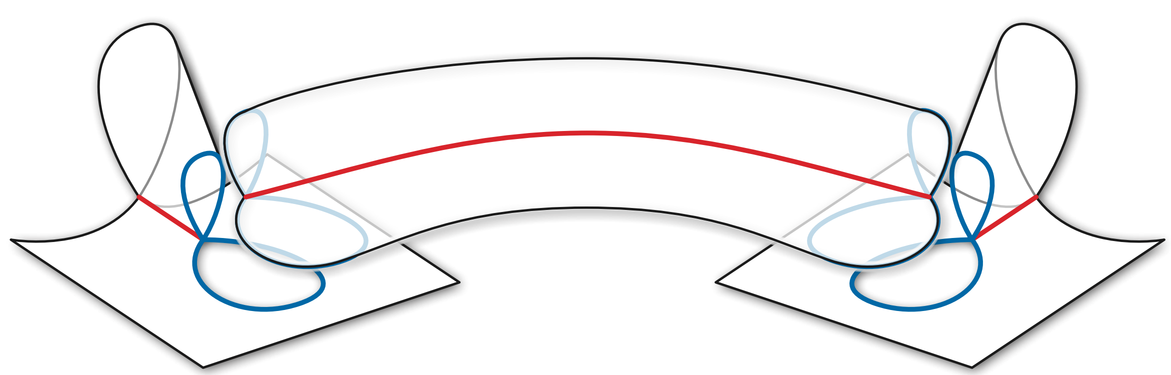
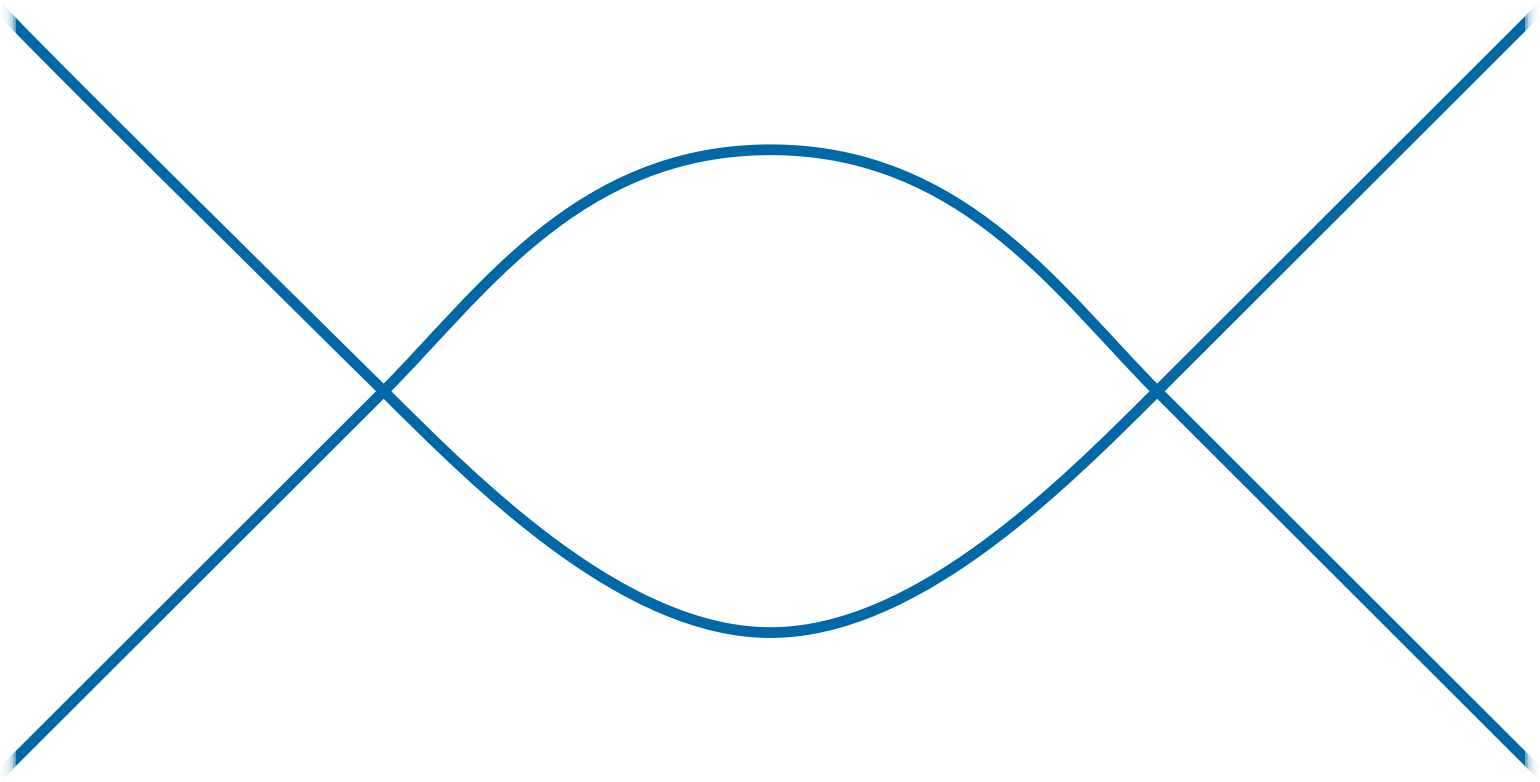


- Since Q^* has an even number of vertices, the number of branch points is even, so we can remove them in pairs. [Hass Hughes 85]



Refining to duals of hex meshes

- “Bubble-wrap” algorithm *[Babson Chan ’08]*



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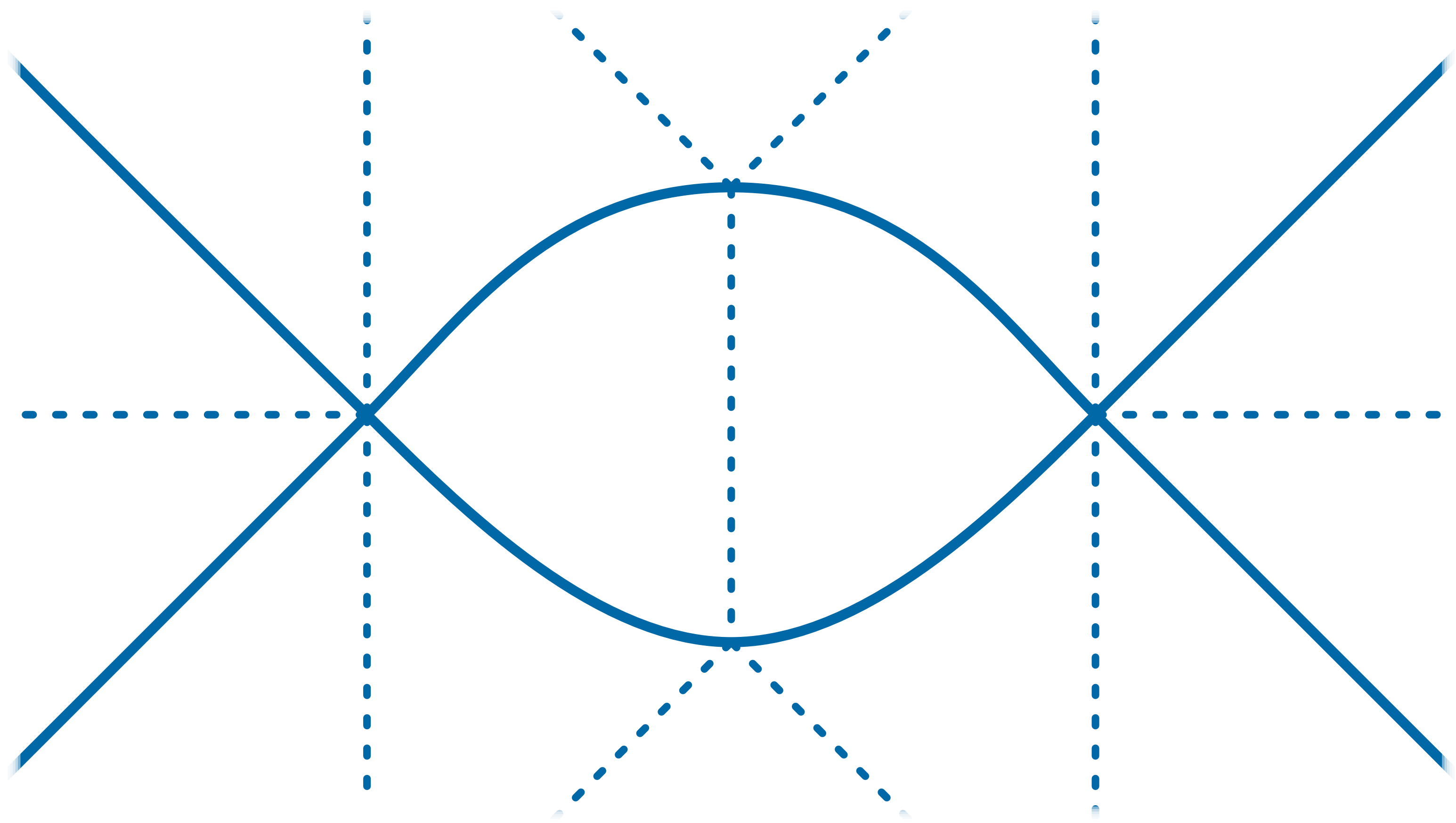

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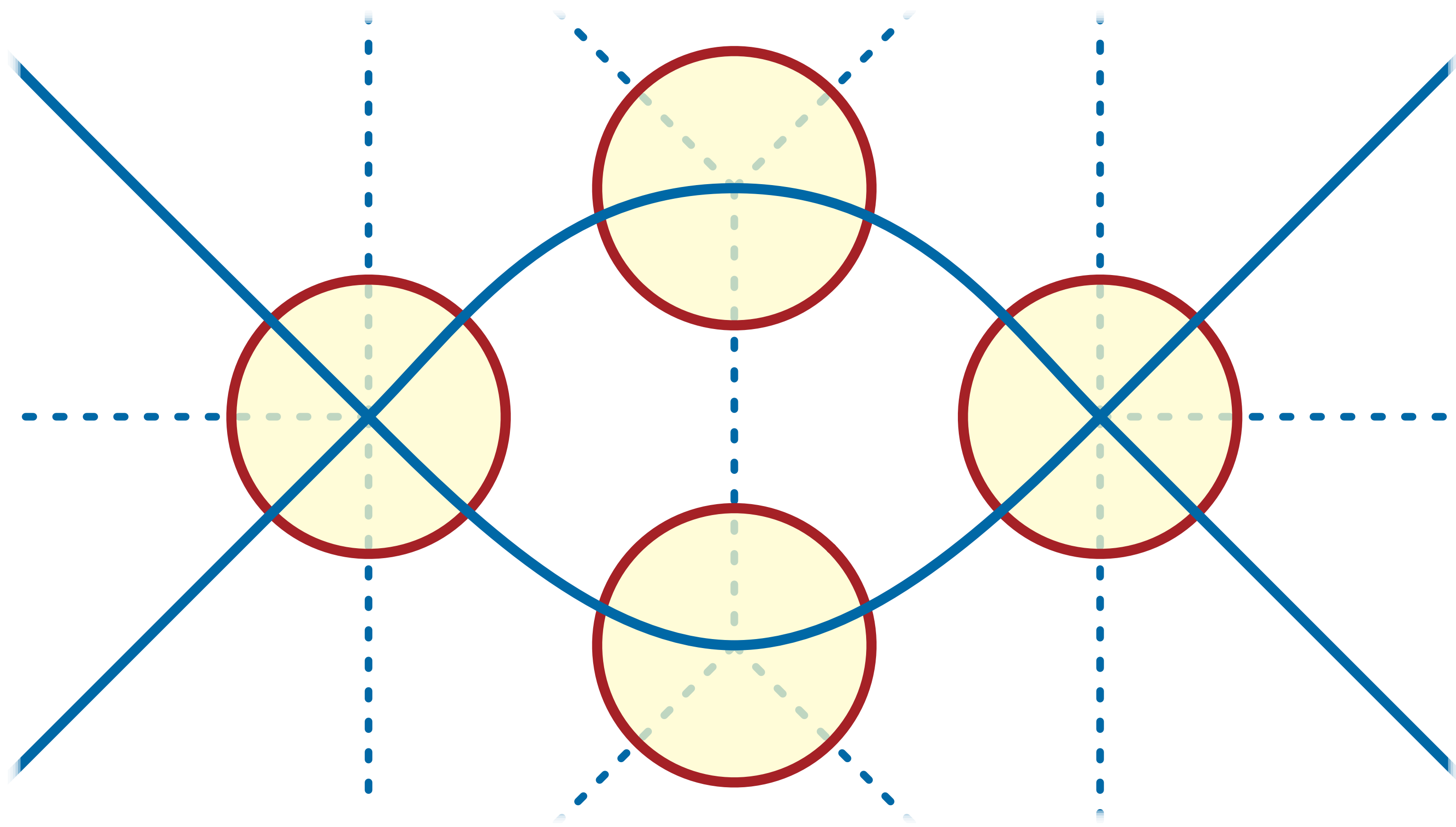

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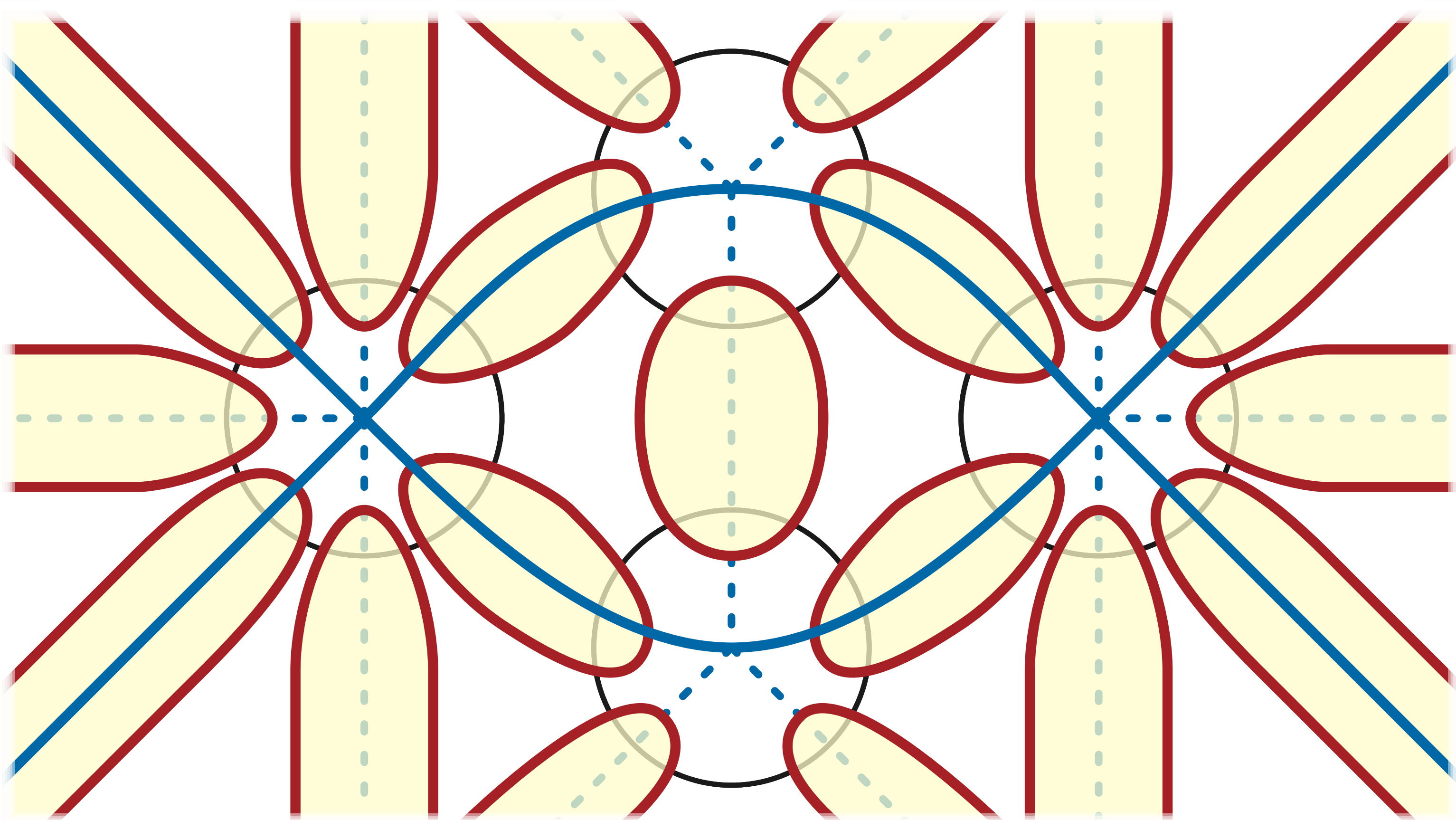

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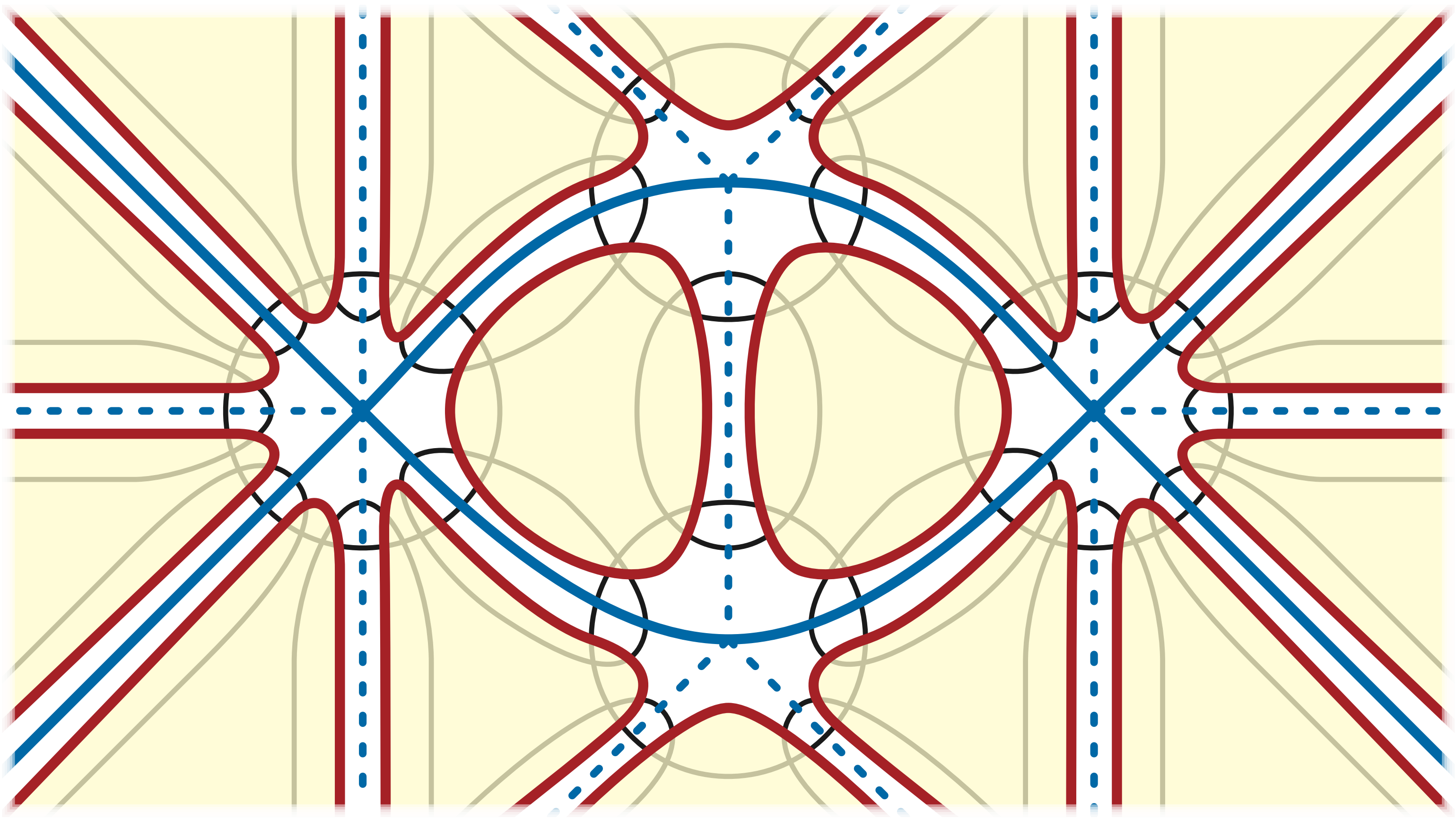

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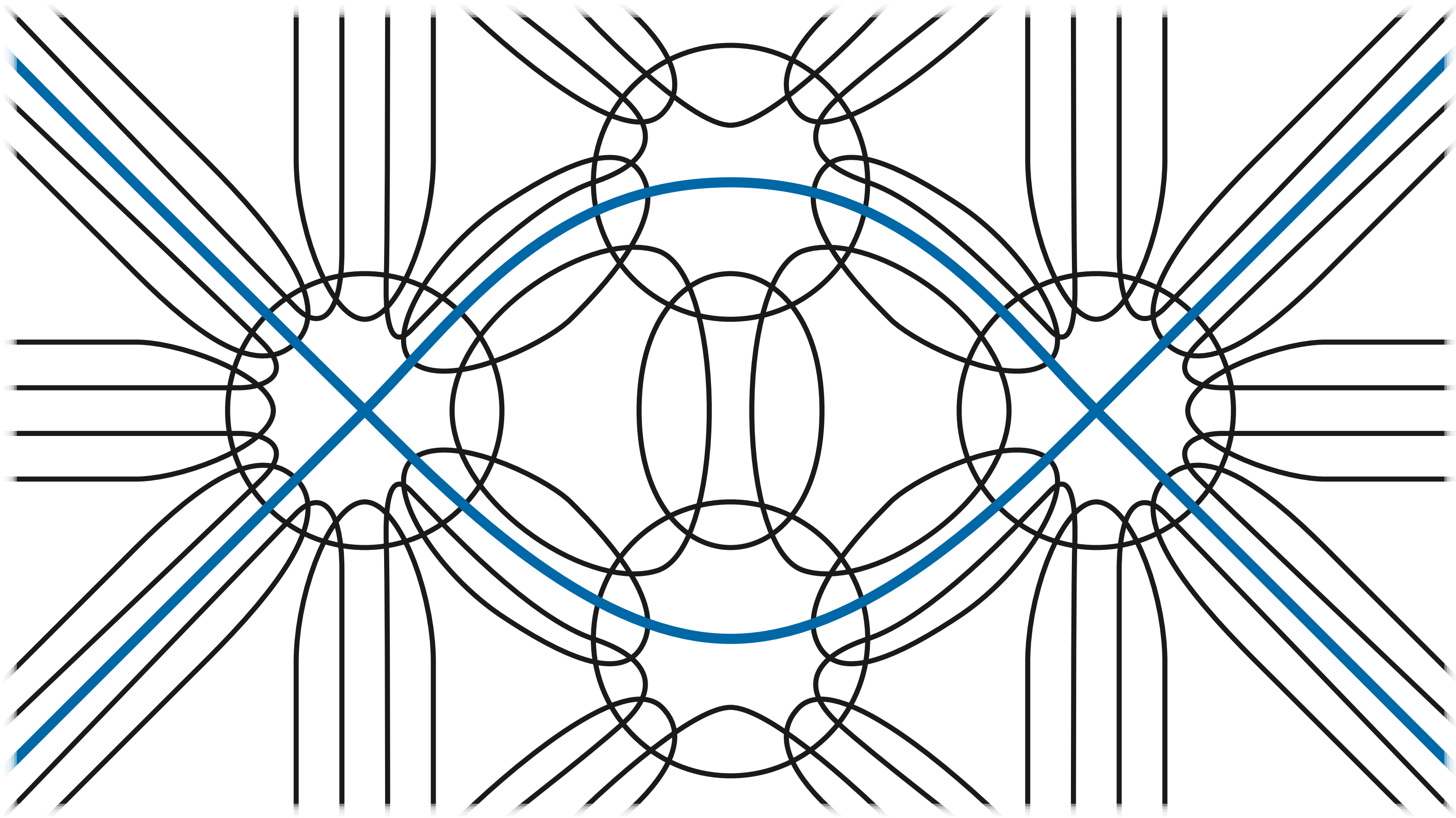

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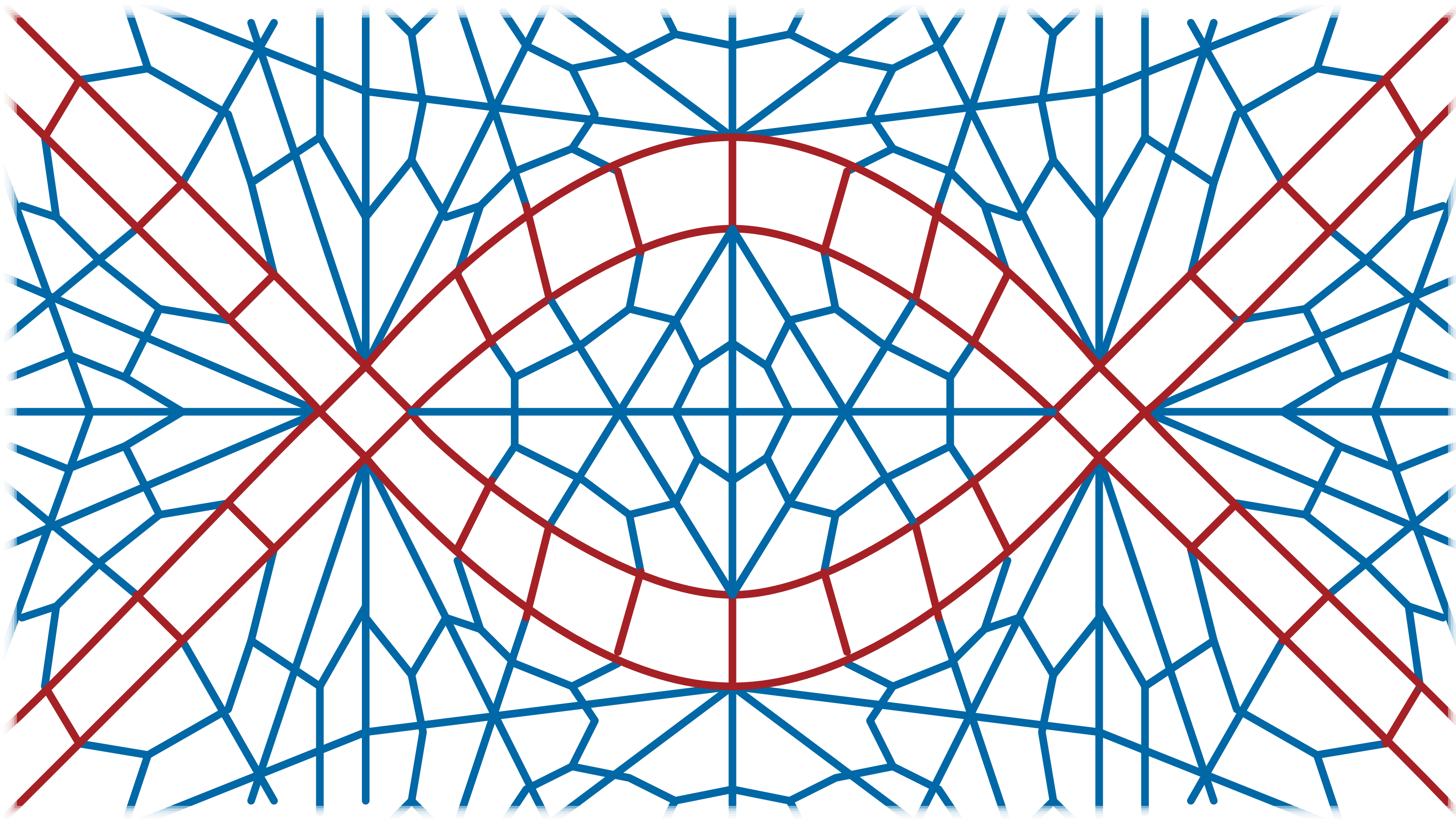


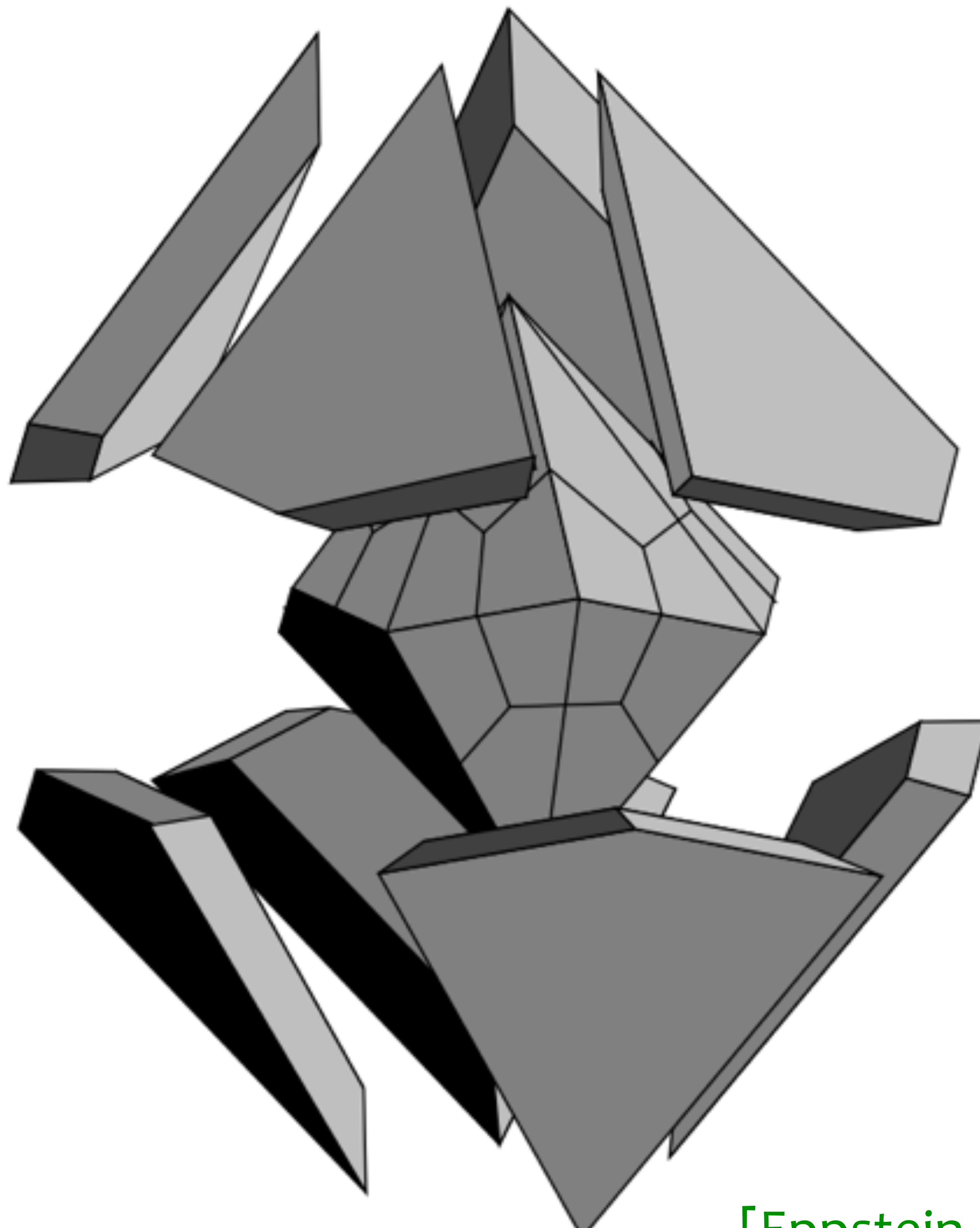
But I’m a computer scientist.

- This proof can be made constructive, but without more effort, there are no good bounds on the complexity of the resulting mesh.
- Effort sucks. Instead, follow Eppstein’s algorithmic proof for bipartite surface meshes.

Algorithm

- Extend Q to a *buffer layer* B of cubes. [Eppstein 99]
- Compute a *triangulation* T of the interior $\Omega \setminus B$.
- *Refine* tetrahedra in T into cubes (using homology of Q^*)
- *Refine* B to match inner and outer boundaries. [Eppstein 99]

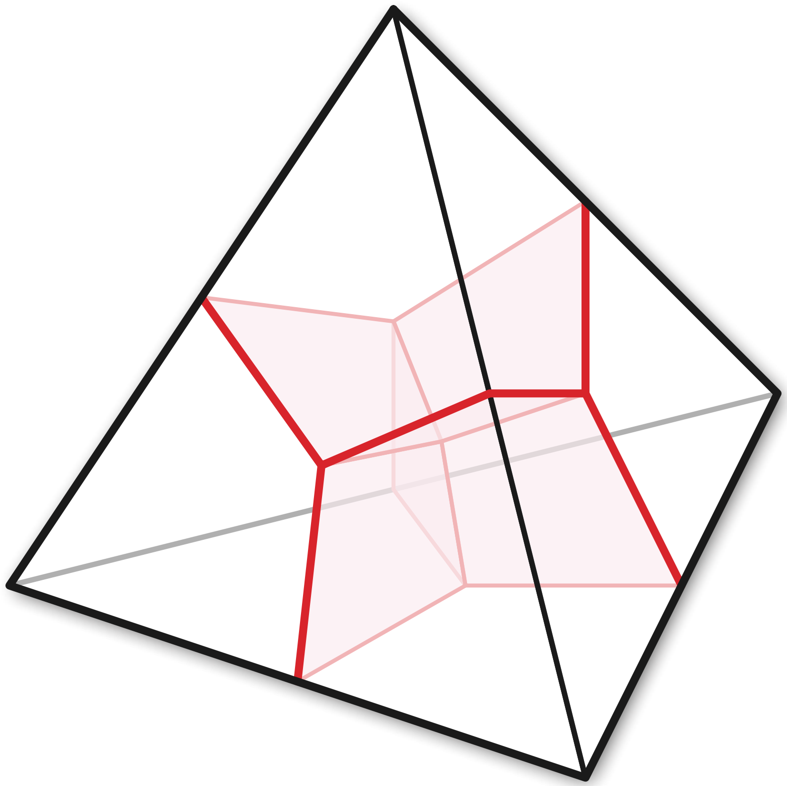
Buffer layer



[Eppstein 99]

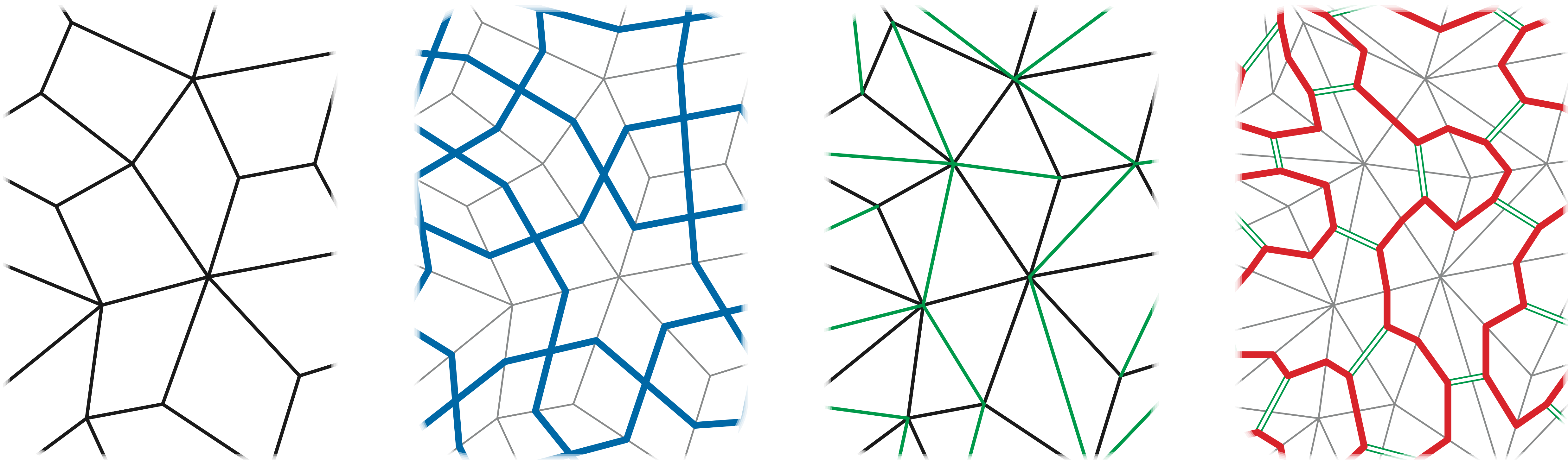
Refining the inner triangulation

- Dual complex *T**: Split each tetrahedron into four cubes, then delete faces of *T*.



Refining the inner triangulation

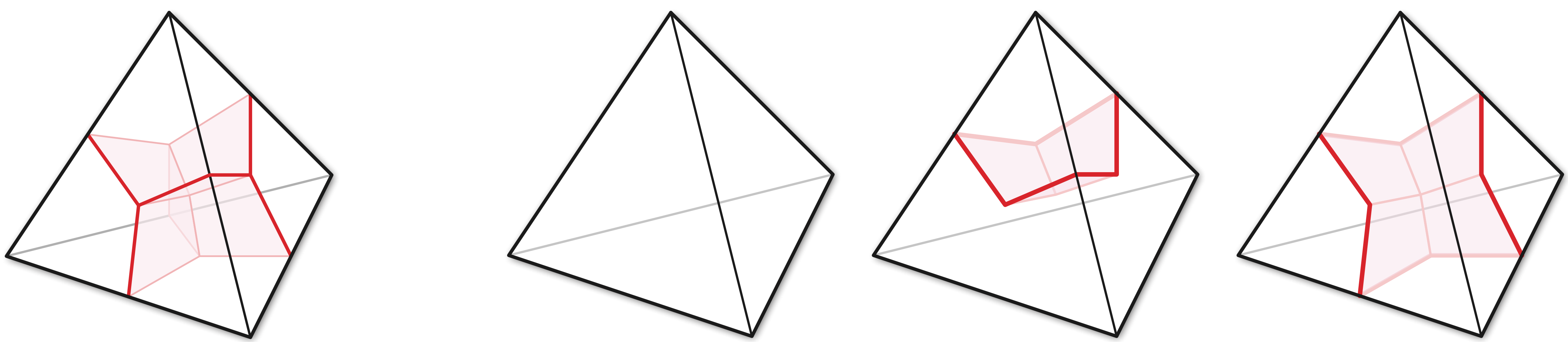
- Find curves C in ∂T^* that are homologous to Q^*



Q | Q^* | $\partial T = Q + \Delta$ | $C = \partial T^* - \Delta^*$

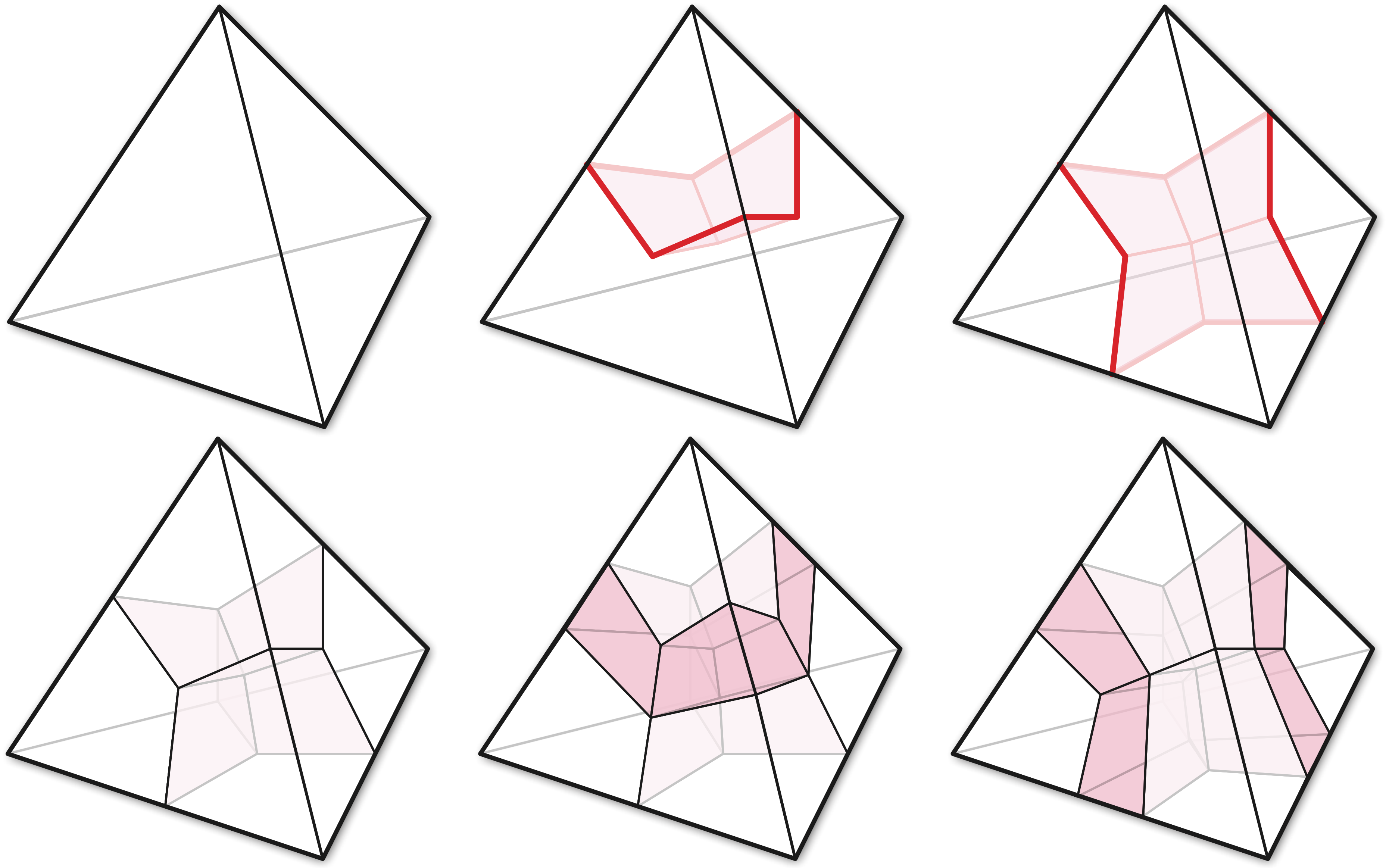
Refining the inner triangulation

- Find an *embedded* surface S in T^* such that $\partial S = C$.
- Solve a system of linear equations over $\mathbf{Z}_2$
  - Variable per facet of T^* = vertex of T
  - Equation per edge of T^* = edge of T
  - If no solution, then Q^* is not null-homologous, so Q is not the boundary of a hex mesh.



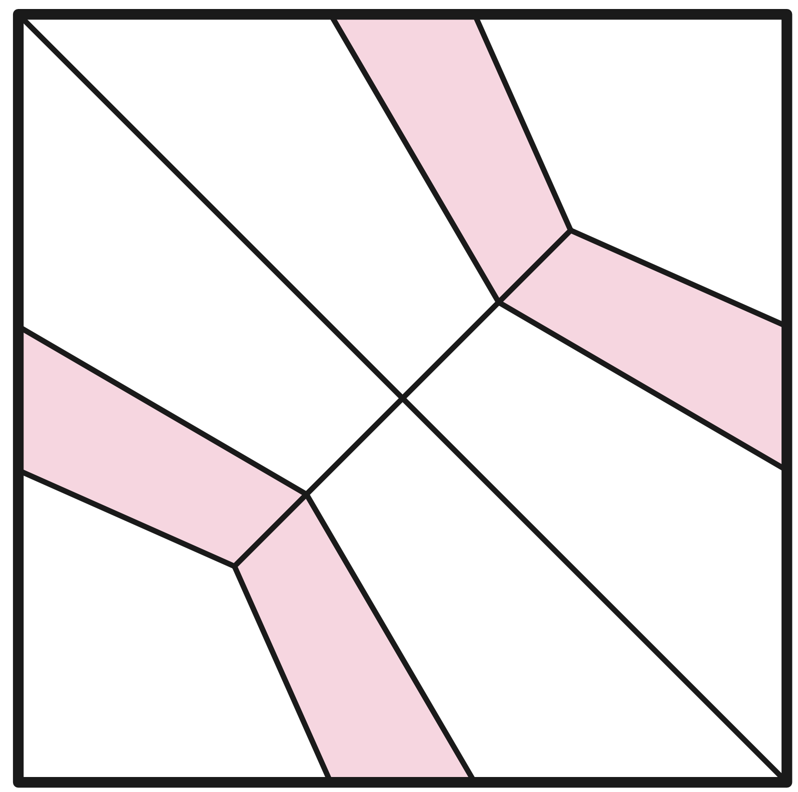
Refining the inner triangulation

- Refine each tetrahedron according to how it intersects S.



Refining the buffer cubes

- Inner face of each buffer cube looks like this:

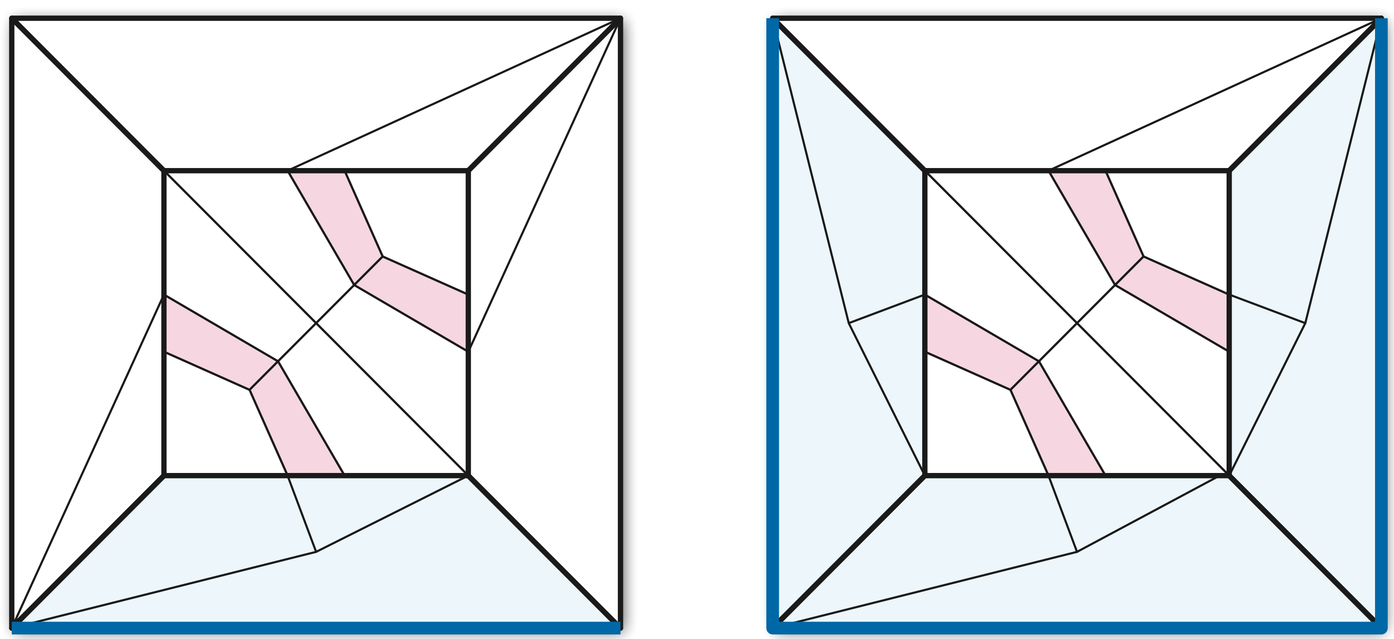


- So "vertical" faces of buffer cubes are hexagons

Refining the buffer cubes

- Let *R* be a subgraph of *Q* such that every quad in *Q* touches an odd number of edges in *R*. [Eppstein 99]

- Refine boundary of each buffer cube into 20 or 22 quads.



- Thurston-Mitchell gives a hex mesh for each buffer cube.

Analysis

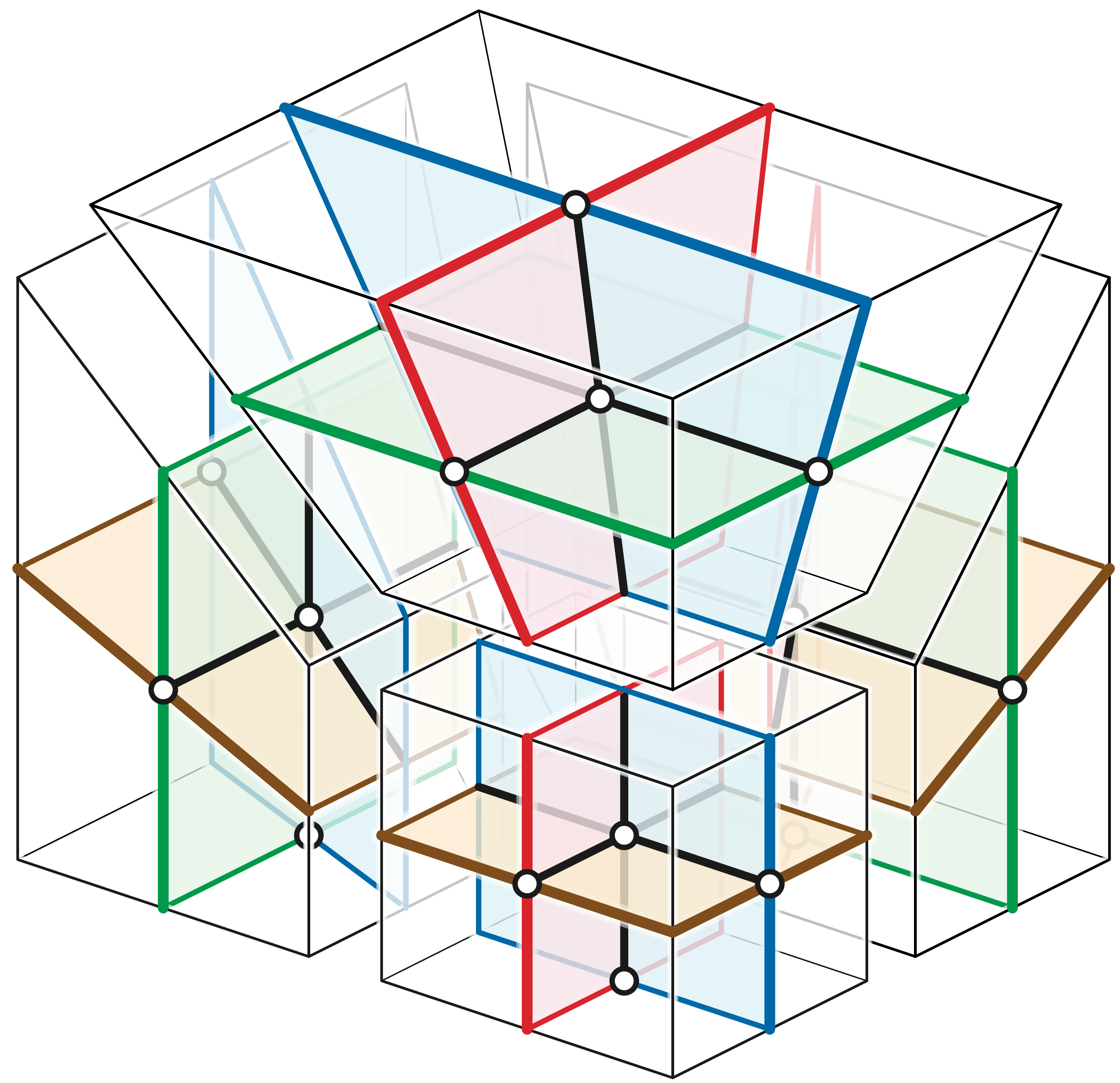
- The number of cubes in the output mesh is a constant times the number of tetrahedra in T.
- Any polyhedron with n quad facets is either the boundary of a topological hex mesh with complexity $O(n^2)$ or is not the boundary of any hex mesh. [Chazelle Shouraboura 95]
- Our algorithm runs in $O(n^6)$ time.
  - We can check whether a topological hex mesh *exists* in $O(g^2n^2)$ time. [Dey Fan Wang SIGGRAPH 2013]
  - Both algorithms are significantly faster in practice.

Main result

Let Ω be a compact subset of $\boldsymbol{R}^3$ whose boundary is a 2-manifold. Let Q be a topological quad mesh of $\partial\Omega$ with an even number of facets. The following are equivalent:

- Q is the boundary of a hex mesh of Ω.
- No odd cycle in Q is null-homologous in Ω.
- Q^* is null-homologous in Ω.

¡Gracias!



http://www.cs.uiuc.edu/~jeffe/pubs/hexmesh.html